Entropy Under Additive Bernoulli and Spherical Noises

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Bernoulli Noise VS Spherical Noise

- Let $Z^n \sim \mathrm{Bernoulli}(\delta)^{\otimes n}$
- Define the δn-Hamming sphere

$$\mathcal{S}_{\delta n,n} \triangleq \{x^n \in \{0,1\}^n \ : \ |x^n| = \delta n\}$$

and let $U^n \sim \mathrm{Unif}(\mathcal{S}_{\delta n,n})$

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$Z_i \stackrel{d}{=} U_i$, so $H(Z^n) > H(U^n)$

In fact, from Stirling's approximation

$$\begin{aligned} H(Z^n) - H(U^n) &= nh(\delta) - \log |\mathcal{S}_{\delta n,n}| \\ &\in \frac{1}{2}\log n + \left[\frac{1}{2}\log(2\pi\delta(1-\delta)), \frac{1}{2}\log(8\delta(1-\delta))\right]. \end{aligned}$$

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This talk

Let $X^n \perp\!\!\!\perp (Z^n, U^n)$ be some RV on $\{0,1\}^n$

What can we say on $H(X^n + Z^n) - H(X^n + U^n)$?

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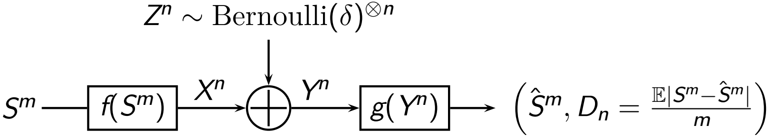
This talk

Let $X^n \perp\!\!\!\perp (Z^n, U^n)$ be some RV on $\{0,1\}^n$

What can we say on $H(X^n + Z^n) - H(X^n + U^n)$?

- Always positive (like for $X^n = 0^n$)?
- How does it scale with n?

Motivation: Finite Blocklength Bounds for JSCC



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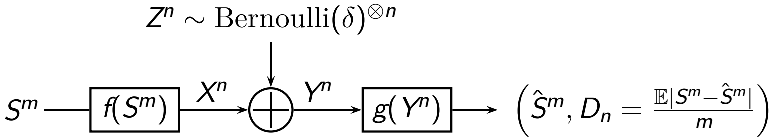

Let $A = \frac{|Z^n| - n\delta}{\sqrt{n}}$
By CLT: "$A \sim \mathcal{N}(0, \delta(1-\delta))$"

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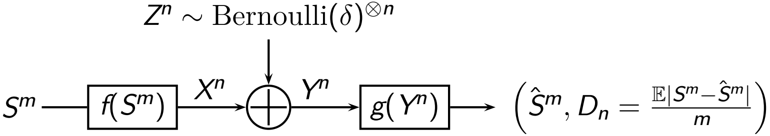

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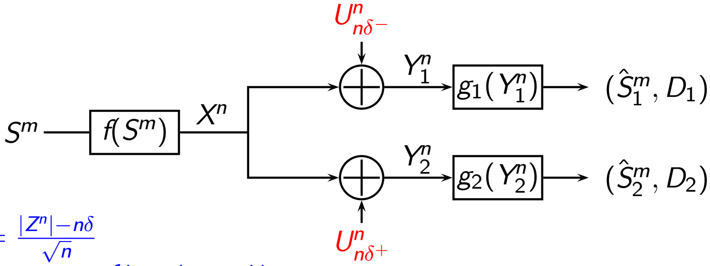

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Set: $\delta^- = \delta - \frac{a}{\sqrt{n}}$, $\delta^+ = \delta + \frac{a}{\sqrt{n}}$

$$D_n = \mathbb{E}[D_n \mid |A| = a] \geq \frac{1}{2}\mathbb{E}\left[D_1(A) + D_2(A)\right]$$

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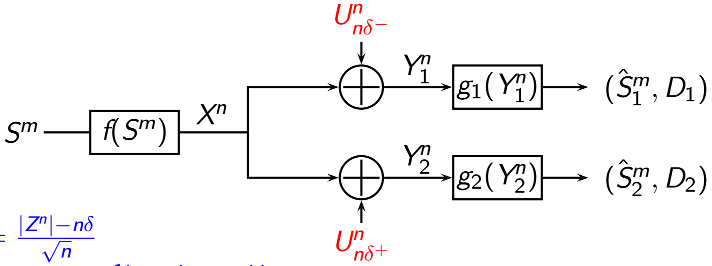

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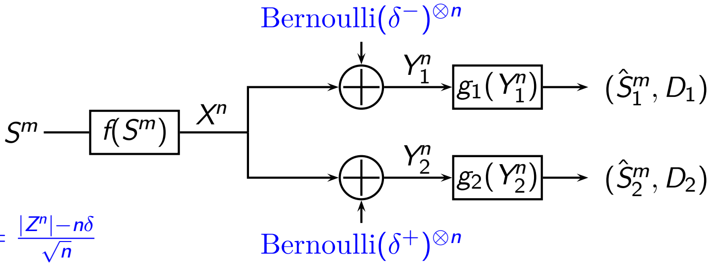
$$D_n = \mathbb{E}[D_n \mid |A| = a] \geq \frac{1}{2}\mathbb{E}\left[D_1(A) + D_2(A)\right]$$

Tradeoff between D_1 and D_2 is hard to analyze for the spherical noise BC

Motivation: Finite Blocklength Bounds for JSCC



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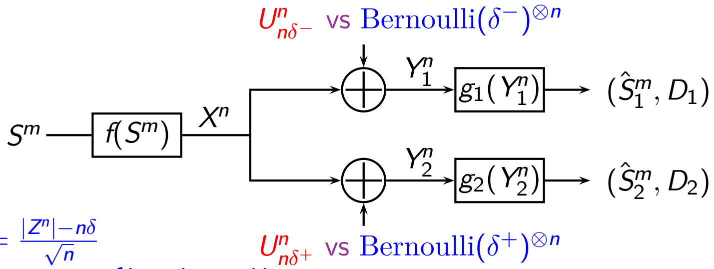
$$D_n = \mathbb{E}[D_n \mid |A| = a] \geq \frac{1}{2}\mathbb{E}\left[D_1(A) + D_2(A)\right]$$

For Bernoulli noise BC, things are easier

Motivation: Finite Blocklength Bounds for JSCC



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$$D_n = \mathbb{E}[D_n \mid |A| = a] \geq \frac{1}{2}\mathbb{E}\left[D_1(A) + D_2(A)\right]$$

But how accurate is the approximation of spherical noise with Bernoulli noise?

Results - Bound on Entropy Difference

Theorem

For any $0 < \delta < 1/2$ we have that

$$c_1(\delta)\sqrt{n} + o(\sqrt{n}) \le \sup_{X^n} H(X^n + U^n) - H(X^n + Z^n) \le \sqrt{2\pi}c_1(\delta)\sqrt{n}$$

where

$$c_1(\delta) = \log\left(\frac{1-\delta}{\delta}\right)\sqrt{\frac{\delta(1-\delta)}{2\pi}}$$

The case $X^n = 0^n$ is non-representative:
$H(X^n + U^n)$ can be greater than $H(X^n + Z^n)$ by $O(\sqrt{n})$, and this is tight

Results - Bound on Entropy Difference

Theorem

For any $0 < \delta < 1/2$ we have that

$$\frac{1}{2}\log n + c_3(\delta) \leq \sup_{X^n} H(X^n + Z^n) - H(X^n + U^n) \leq c_2(\delta)\sqrt{n}$$

where

$$c_2(\delta) = 4\log\left(\frac{1}{\delta}\right)\sqrt{\frac{h(\delta)(1-\delta)}{\delta}}$$
$$c_3(\delta) = \frac{1}{2}\log\left(2\pi\delta(1-\delta)\right)$$

Lower bound is trivially achieved by $X^n = 0^n$. Upper bound is challenging

Results - Bound on Entropy Difference

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For any $0 < \delta < 1/2$ we have that

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$$c_2(\delta) = 4\log\left(\frac{1}{\delta}\right)\sqrt{\frac{h(\delta)(1-\delta)}{\delta}}$$

Follows from combining the two previous results

$\sup_{X^n} |H(X^n + U^n) - H(X^n + Z^n)|$ is fully characterized, up to a multiplicative constant

Results - Spherical MGL

Theorem (Spherical MGL)

For any $0 < \delta < 1/2$ and any RV X^n on $\{0,1\}^n$, we have that

$$H(X^n + U^n) \geq nh\left(\delta * h^{-1}\left(\frac{H(X^n)}{n}\right)\right) - 8\frac{1-\delta}{\delta}\log n.$$

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Wyner and Ziv's MGL gives

$$H(X^n + Z^n) \geq nh\left(\delta * h^{-1}\left(\frac{H(X^n)}{n}\right)\right).$$

Thus, MGL and the spherical MGL only differ by an $O(\log n)$ additive term

Some snippets of the proofs

Warm Up Bound via Coupling

Polyanskiy-Wu'16:
For any P_{U^n,Z^n} with marginals P_{U^n}, P_{Z^n} (coupling)

$$
\begin{aligned}
H(X^n + U^n) - H(X^n + Z^n) &\le H(X^n + U^n, X^n + Z^n) - H(X^n + Z^n) \\
&= H(X^n + U^n | X^n + Z^n) \\
&\le H(U^n + Z^n) \\
&\le nh\left(\frac{\mathbb{E}|U^n + Z^n|}{n}\right)
\end{aligned}
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similarly, $H(X^n + Z^n) - H(X^n + U^n) \leq nh\left(\frac{\mathbb{E}|U^n+Z^n|}{n}\right)$

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Draw a RV $W \sim \mathrm{Binomial}(n, \delta)$ and a uniform permutation Π

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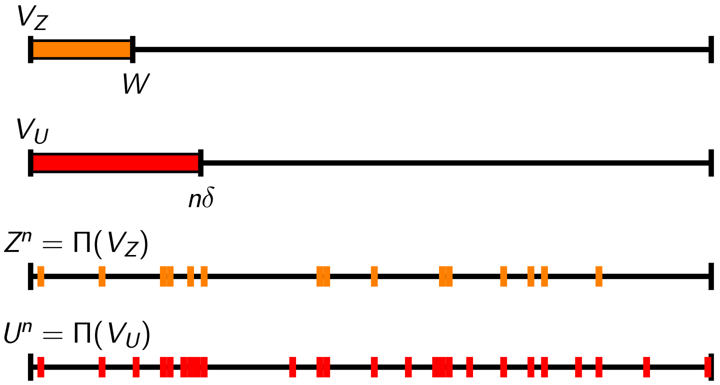

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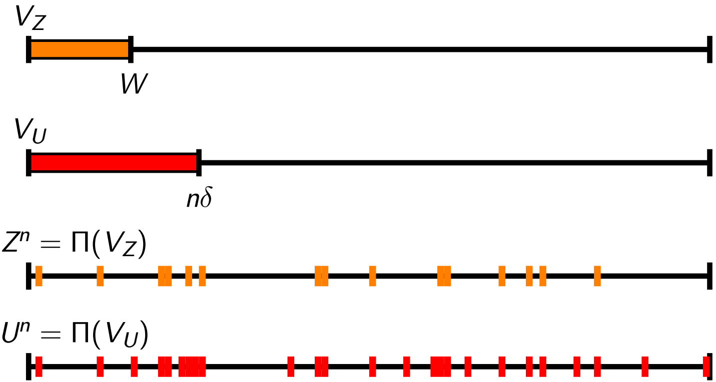

$$\mathbb{E}|Z^n + U^n| = \mathbb{E}|W - \delta n| \leq \sqrt{n\delta(1-\delta)}$$

Warm Up Bound via Coupling

For any P_{U^n,Z^n} with marginals P_{U^n}, P_{Z^n} (coupling)

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Lemma

For any X^n on $\{0,1\}^n$

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Got $O(\sqrt{n}\log n)$, want $O(\sqrt{n})$

One-Sided Improvement via Coupling

- Let $P = P_{X^n+U^n}$, $Q = P_{X^n+Z^n}$

$$
\begin{aligned}
H(X^n + U^n) - H(X^n + Z^n) &= \mathbb{E}\left[\log \frac{Q(X^n + Z^n)}{P(X^n + U^n)}\right] \\
&= \mathbb{E}\left[\log \frac{Q(X^n + Z^n)}{Q(X^n + U^n)} \frac{Q(X^n + U^n)}{P(X^n + U^n)}\right] \\
&= \mathbb{E}\left[\log \frac{Q(X^n + Z^n)}{Q(X^n + U^n)}\right] - D(P \| Q)
\end{aligned}
$$

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Using the coupling introduced in previous slide gives
$\mathbb{E}|Z^n + U^n| \le \sqrt{n\delta(1-\delta)}$

One-Sided Improvement via Coupling

Lemma

For any $0 < \delta < 1/2$ we have that

$$sup_{X^n} H(X^n + U^n) - H(X^n + Z^n) \leq \sqrt{2\pi} c_1(\delta)\sqrt{n}$$

where

$$c_1(\delta) = \log\left(\frac{1-\delta}{\delta}\right)\sqrt{\frac{\delta(1-\delta)}{2\pi}}$$

Random Coding Lower Bound on $H(X^n + U^n) - H(X^n + Z^n)$

- By random coding, we can show that the channel
  $x^n \mapsto x^n + U^n$ has zero dispersion
  $\Rightarrow \exists \mathcal{C}$ with rate $R = \log 2 - h(\delta) - \frac{\log n}{n}$ and $P_e < n^{-1}$

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Let $X^n \sim \mathrm{Unif}(\mathcal{C})$. By Fano's inequality

$$
\begin{aligned}
H(X^n + U^n) &= I(X^n + U^n; X^n) + H(U^n) \\
&= \underbrace{H(X^n)}_{nR} + \underbrace{H(U^n)}_{nh(\delta) - \frac{1}{2}\log n + const} - \underbrace{H(X^n | X^n + U^n)}_{const} \\
&= n \log 2 - \frac{3}{2} \log n - const
\end{aligned}
$$

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Let $W = |Z^n|$. We have

$$
\begin{aligned}
H(X^n + Z^n) &= H(X^n + Z^n | W) + I(W; X^n + Z^n) \\
&\leq H(X^n + Z^n | W) + \log n \\
&\leq \mathbb{E}_W \left(H(X^n, Z^n | W = w) \bigwedge n \log 2 \right) + \log n \\
&= \mathbb{E}_W \left(H(X^n) + H(Z^n | W = w) \bigwedge n \log 2 \right) + \log n \\
&\leq \mathbb{E}_W \left(H(X^n) + nh\left(\frac{W}{n}\right) \bigwedge n \log 2 \right) + \log n \\
&\leq n \log 2 - c_1(\delta)\sqrt{n}(1 + o(1))
\end{aligned}
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Random Coding Lower Bound on $H(X^n + U^n) - H(X^n + Z^n)$

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lemma

For any $0 < \delta < 1/2$ we have that

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where

$$c_1(\delta) = \log\left(\frac{1-\delta}{\delta}\right)\sqrt{\frac{\delta(1-\delta)}{2\pi}}$$

Upper Bound on $H(X^n + Z^n) - H(X^n + U^n)$

- We look at m-projections of U^n
- We show, by induction, that for every $1 \leq m \leq n$

$$H(X^m + Z^m) - H(X^m + U^m) \leq c(\delta) \sum_{2 \leq k \leq m} \frac{1}{\sqrt{n - (k-1)}}$$

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Let $V^{m-1} = X^{m-1} + U^{m-1}$. From standard chain rule tricks

$$\begin{aligned} H(X^m + Z^m) - H(X^m + U^m) = {} & H(X^{m-1} + Z^{m-1}) - H(X^{m-1} + U^{m-1}) \\ & + H(X_m + Z_m | V^{m-1}) - H(X_m + U_m | V^{m-1}) \end{aligned}$$

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Bound first term using induction Hypothesis
Challenge in bounding second term is that $X_m \not\perp\!\!\!\perp U_m | V^{m-1}$

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Let $V^{m-1} = X^{m-1} + U^{m-1}$. From standard chain rule tricks

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$$+ H(X_m + Z_m | V^{m-1}) - H(X_m + U_m | V^{m-1})$$

Upper Bound on $H(X^n + Z^n) - H(X^n + U^n)$

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Lemma: Let A, B, C be RVs, and $\bar{B} \stackrel{d}{=} B$, $\bar{B} \perp\!\!\!\perp (A, B, C)$, then

$$H(A + \bar{B} | C) - H(A + B | C) \leq \gamma \sqrt{I(A, C; B)}$$

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$$I(X_m, V^{m-1}; U_m) = I(X_m, X^{m-1} + U^{m-1}; U_m) \leq I(U^{m-1}; U_m)$$

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$$H(X^m + Z^m) - H(X^m + U^m) \leq c(\delta) \sum_{2 \leq k \leq m-1} \frac{1}{\sqrt{n-(k-1)}} + c'(\delta) c''(\delta) \frac{1}{\sqrt{n-(m-1)}}$$

Can be shown that $I(U^{m-1}; U_m) \leq c''(\delta) \frac{1}{n-(m-1)}$

Upper Bound on $H(X^n + Z^n) - H(X^n + U^n)$

- We look at m-projections of U^n
- We show, by induction, that for every $1 \leq m \leq n$

$$H(X^m + Z^m) - H(X^m + U^m) \leq c(\delta) \sum_{2 \leq k \leq m} \frac{1}{\sqrt{n - (k-1)}}$$

lemma

For any $0 < \delta < 1/2$ we have that

$$sup_{X^n} H(X^n + Z^n) - H(X^n + U^n) \leq c_2(\delta)\sqrt{n}$$

where

$$c_2(\delta) = 4 \log \left(\frac{1}{\delta}\right) \sqrt{\frac{h(\delta)(1-\delta)}{\delta}}$$

Spherical MGL

Define the RV $A_m = \Pr(U_m = 1|U^{m-1})$. Can be shown that $\mathbb{E}A_m = \delta$ and $\mathbb{E}(A_m - \delta)^2 \leq \frac{\delta(1-\delta)}{n-m}$. We have

$$
\begin{aligned}
H(X^n + U^n) &= \sum_{m=1}^{n} H(X_m + U_m | X^{m-1} + U_1^{m-1}) \\
&\geq \sum_{m=1}^{n} H(X_m + U_m | X^{m-1}, U^{m-1}) \\
&= \sum_{m=1}^{n} \mathbb{E}h\left(\Pr(X_m = 1|X^{m-1}) * \Pr(U_m = 1|U^{m-1})\right) \\
&\geq \sum_{m=1}^{n} \mathbb{E}h\left(h^{-1}\left(H(X_m|X^{m-1})\right) * A_m)\right)
\end{aligned}
$$

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&\ge \sum_{m=1}^{n} \mathbb{E}h\left(h^{-1}\left(H(X_m | X^{m-1})\right) * A_m\right)
\end{aligned}$$

Now, just use the fact that A_m is concentrated around δ

Spherical MGL

Theorem (Spherical MGL)

For any $0 < \delta < 1/2$ and any RV X^n on $\{0,1\}^n$, we have that

$$H(X^n + U^n) \geq nh\left(\delta * h^{-1}\left(\frac{H(X^n)}{n}\right)\right) - 8\frac{1-\delta}{\delta}\log n.$$

Conclusions

- The difference between $H(X^n + Z^n)$ and $H(X^n + U^n)$ was studied
- $\sup_{X^n} H(X^n + U_n) - H(X^n + Z^n) = c_1(\delta)\sqrt{n}$
- $\sup_{X^n} H(X^n + Z_n) - H(X^n + U^n) \in [\frac{1}{2}\log n + c_3(\delta), c_2(\delta)\sqrt{n}]$
- $H(X^n + U^n) \geq \mathrm{MGL}(H(X^n)) - c_4(\delta)\log n$
- The channels $x^n \mapsto x^n + U^n$ and $x^n \mapsto x^n + Z^n$ are not ordered in "less noisy" sense
- We currently don't know if $x^n \mapsto x^n + U^n$ is "more capable" than $x^n \mapsto x^n + Z^n$