Cyclic-Coded Integer-Forcing Equalization

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***Abstract*—A discrete-time intersymbol interference (ISI) channel with additive Gaussian noise is considered, where only the receiver has knowledge of the channel impulse response. An approach for combining decision-feedback equalization with channel coding is proposed, where decoding precedes the removal of ISI. The proposed approach involves equalizing the channel impulse response to a response with integer-valued coefficients in conjunction with utilizing cyclic block codes. Leveraging the property that a cyclic code is closed under cyclic integer-valued convolution allows us to perform decoding prior to applying decision feedback. Explicit bounds on the performance of the proposed scheme are derived.**

***Index Terms*—Combined equalization and coding, cyclic codes, decision-feedback equalization (DFE), intersymbol interference (ISI), linear Gaussian channels, single-carrier modulation.**

I. Introduction

THE discrete-time intersymbol interference (ISI) channel with additive Gaussian noise is one of the most basic channel models arising in digital communications. Thus, considerable effort has been devoted to developing effective transmission schemes for this channel; see, e.g., [1], for a comprehensive survey. The channel is described by

$$\begin{aligned} y_k &= h_0 x_k + \sum_{m \neq 0} h_m x_{k-m} + n_k \\ &= h_0 x_k + \mathrm{ISI}_k + n_k \end{aligned} \tag{1}$$

where ISI_k is the ISI resulting from other data symbols, and n_k is additive white Gaussian noise (AWGN) with zero mean and unit power.

The channel model may further be characterized by the availability of channel state information (CSI), where we distinguish between the case where CSI is available to the transmitter and the receiver alike and the case where CSI is available to the receiver only. As we next briefly recall, while the distinction between these two cases does not make a great difference at high signal-to-noise ratios (SNR) (which is the main focus of this paper) in terms of capacity [2] (i.e., whether water-filling may be performed or not), it is of significant consequence for the design and implementation of equalization and coding schemes.

In the past decades, coding for AWGN channels has reached an advanced state, and practical coding schemes (e.g., turbo and LDPC codes) operating near capacity are known. It is thus desirable to combine AWGN coding and decoding techniques with equalization in a *modular* way, with the aim of approaching the capacity of the ISI channel.

The multitude of approaches developed to achieve reliable communication over the ISI channel may be roughly divided into two classes: multicarrier approaches and single-carrier approaches. Both approaches may in principle be used to operate at rates close to the capacity of the ISI channel, but offer different practical tradeoffs as we briefly touch upon next.

In multicarrier transmission, the ISI channel is transformed into a set of parallel AWGN subchannels, each subchannel corresponding to a different frequency bin and experiencing a different SNR. This approach has the advantage that the subchannels are ISI free, and thus the problems of equalization and decoding are decoupled. However, it has some drawbacks: the alphabet of the transmitted symbols is essentially continuous, which in turn makes the approach inapplicable to some media. A related and even more restricting phenomenon associated with multicarrier transmission is that it results in a high peak-to-average power ratio which may also be undesirable (see, e.g., [3], [4]). Furthermore, when CSI is available only at the receiver, bit allocation is precluded, and channel coding and decoding become more difficult, due to the variation of the SNR across subchannels.

Single-carrier approaches try to eliminate most of the ISI without severely increasing noise power. The simplest approach is that of linear equalization (LE), consisting only of a "feed-forward" equalizer (FFE), which roughly transforms the channel into an additive colored Gaussian noise channel, where the minimum mean-squared error (MMSE) criterion corresponds to (linearly) maximizing the signal-to-interference-plus-noise (SINR) at the "slicer". Performance may be improved using (nonlinear) decision-feedback equalization (DFE), in addition to FFE. Specifically, MMSE-DFE will be discussed in greater detail in Section II. In fact, as shown by Guess and Varanasi [5], the MMSE-DFE architecture is optimal in the sense of attaining mutual information, and allows us to approach capacity with AWGN encoding/decoding, if decisions (fed to the DFE) are based on codewords rather than symbols. See also [6]–[8].

Unfortunately, the Guess–Varanasi approach, while quite pleasing from a theoretical perspective, requires long interleaving as well as long zero padding, which in turn incurs long latency. This drawback can be avoided if CSI is available at

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the transmitter by employing Tomlinson–Harashima precoding ([9], [10], see also [2]), which essentially moves the DFE to the transmitter. However, precoding is inapplicable if the transmitter has no knowledge of the channel.

The approach proposed in this paper allows us to avoid error propagation *without* incorporating an interleaver and requires CSI to be available at the receiver only. In essence, the proposed method enables us to perform block decoding *before* decision feedback is performed.

We build on the integer-forcing (IF) equalization approach, which was recently proposed [11] in the context of general MIMO channels with CSI available at a receiver only. In this approach, multiple streams are encoded using an identical linear code and the receiver equalizes the channel matrix to any full-rank integer matrix rather than to the identity.

In the context of IF equalization, an ISI channel may be viewed as a Toeplitz matrix. This special structure allows us to further replace the multiple codewords (in the context of ISI, this number would be large) with a *single* codeword provided that the linear code is further a *cyclic code*.[1]

This paper is organized as follows. Section II recalls some basic results on single-carrier equalization. Section III describes IF equalization as well as the general architecture of the proposed scheme. Section IV derives the criterion for choosing the FFE. Section V derives bounds on the noise enhancement induced by the FFE. In Section VI, the performance of the scheme is analyzed, and some examples are given. Section VII discusses practical coding techniques for IF equalization at high transmission rates. This paper concludes with Section VIII.

II. Preliminaries

We briefly review basic single-carrier equalization architectures. In the sequel, we use D-transform notation for sequences; i.e., a sequence $\{s_k\}$ is represented by its D-transform $S(D) = \sum_k s_k D^k$. For example, the channel (1) may be expressed as

$$Y(D) = H(D)X(D) + N(D).$$

The simplest criterion for LE is that of zero forcing (ZF), where the ISI is completely canceled using an FFE only. This corresponds to taking the front-end (linear) filter to be

$$A_{\text{ZF}}(D) = \frac{1}{H(D)}$$

resulting in the equalized channel response $G(D) = 1$. The induced noise enhancement can be large, especially when $H(D)$ has zeros near the unit circle. A variant that takes into account both ISI and noise enhancement is the linear MMSE equalizer

$$A_{\text{MMSE-LE}}(D) = \frac{H^*(D^{-*})}{H(D)H^*(D^{-*}) + 1/\mathsf{SNR}}.$$

The MMSE-LE suffers from smaller (and in particular bounded) noise enhancement while allowing some residual ISI. The MMSE criterion is equivalent to maximizing the SINR at the slicer [7].

[1]The crucial element needed is that the code is linear and shift invariant, not necessarily cyclic.

DFE (see Fig. 1) is based on using previously detected symbols in order to cancel the induced ISI from the symbol entering the slicer. In this approach, if all previous data symbols are detected without error, then postcursor ISI can be removed. Specifically, the output of the FFE in Fig. 1 is given by

$$\begin{aligned} y'_k &= x_k * g_k + z_k \\ &= x_k + \sum_{m=-\infty}^{-1} x_{k-m} g_m + \sum_{m=1}^{\infty} x_{k-m} g_m + z_k \\ &= x_k + \text{ISI}_k^{\text{pre}} + \text{ISI}_k^{\text{post}} + z_k \end{aligned}$$

where $G(D)$ is the equivalent channel after the operation of the FFE and where without loss of generality we have assumed that the FFE normalizes g_0 to equal 1. The DFE then subtracts the term $\hat{\text{ISI}}_k^{\text{post}} = \sum_{m=1}^{\infty} \hat{x}_{k-m} g_m$ from y'_k, where $\hat{x}_k$ are decisions on past transmitted symbols, giving rise (assuming correct past decisions) to the equivalent channel

$$y''_k = x_k + \text{ISI}_k^{\text{pre}} + z_k. \tag{2}$$

When using the optimal FFE and DFE, the induced channel (2) is an additive white noise channel with the same capacity (assuming the input x_k is i.i.d. Gaussian) as that of the discrete ISI channel (1) (see [7]). Combining DFE with coding, however, is a nontrivial task. Since a decision on the value of the last symbol $\hat{x}_k$ must enter the feedback loop at every time instance, there is an intrinsic tension with the latency required for channel coding. Many approaches have been suggested in order to overcome this obstacle (see, for example, [12]–[18]), but to the best of our knowledge none of them allow us to exchange the order of decoding and ISI removal which is the aim of this study. Doing so directly addresses the basic problem of ensuring that reliable decisions enter the DFE loop.

III. Combining Cyclic Codes With IF Equalization

We begin this section with a high-level overview of the proposed IF equalization scheme in the context of ISI channels. The crucial element of the scheme is using a code with a structure that matches that of the channel. Specifically, we would like to use codebooks that are closed under convolution with an integer-valued filter. Rather than pursuing this avenue directly, we utilize the class of extensively studied cyclic codes which are closed under *cyclic* convolution with an integer-valued filter reduced *modulo* some constant.

Definition 1: A linear block code $\mathcal{C}$ of length N over the ring $\mathbb{Z}_q$ is called cyclic, if for every codeword $\mathbf{c} \in \mathcal{C}$, all cyclic shifts of $\mathbf{c}$ are also codewords in $\mathcal{C}$.

Define the modulo operation

$$x \text{mod} q = x - m \cdot q$$

where m is the unique integer such that $x - m \cdot q \in [0, q)$. In this paper, we identify the elements of $\mathbb{Z}_q$ with the points $\{0, 1, \ldots, q-1\} \in \mathbb{R}$, and the arithmetic operations of $\mathbb{Z}_q$ with real addition and multiplication modulo-q.

The linearity of the cyclic code $\mathcal{C}$ implies that any sum modulo-q of two codewords is a codeword. As a consequence,

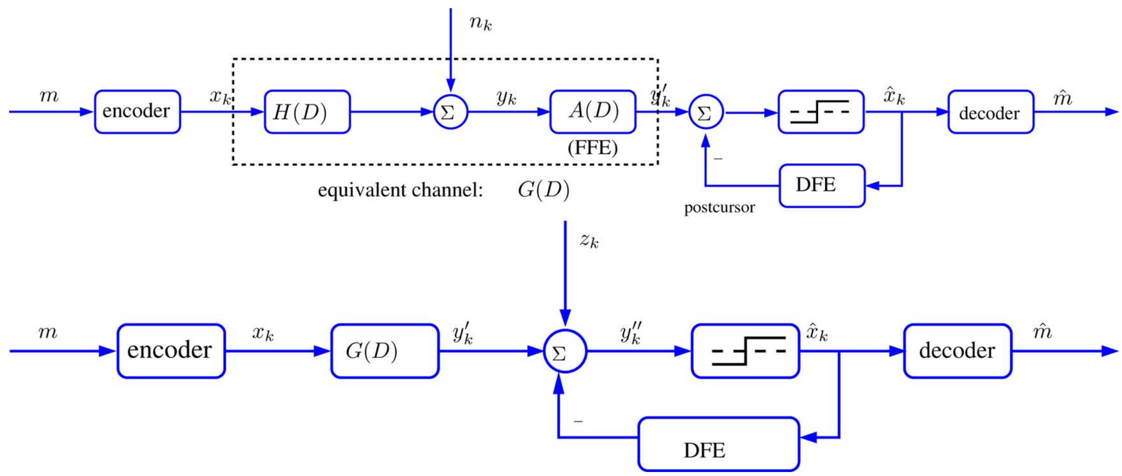

Fig. 1. Decision-feedback equalization.

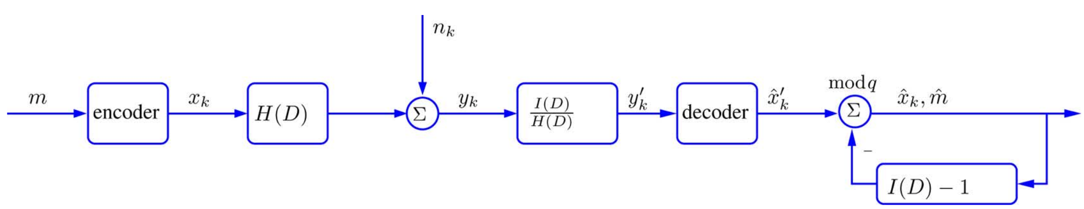

Fig. 2. High-level schematic description of an IF-DFE system (assuming $i_0 = 1$).

when a codeword is multiplied modulo-q by an integer, the result is also a codeword from the same code. The cyclic property of the code ensures that any cyclic shift of a codeword is also a codeword. Since a cyclic convolution modulo-q of a codeword with an integer-valued filter is nothing more than a sum modulo-q of cyclic shifts of the codeword multiplied modulo-q by integers, the result is a codeword from the cyclic code. We denote cyclic convolution w.r.t. block length N by $\otimes$. The following proposition expresses the closure property of a cyclic code w.r.t. cyclic convolution modulo-q.

Proposition 1: Let $\mathcal{C}$ be a cyclic code of length N over $\mathbb{Z}_q$. Then for any vector $\mathbf{i}$ of length N with integer entries

$$[\mathcal{C} \otimes \mathbf{i}] \bmod q \subseteq \mathcal{C}.$$

That is, $\mathcal{C}$ is closed under integer-valued cyclic convolution modulo-q.

Remark 1: Note that Proposition 1 heavily relies on the identity between addition and multiplication in $\mathbb{Z}_q$ and addition and multiplication of integers (over the reals) modulo-q. For nonprime Galois fields such a relation does not hold, and therefore cyclic codes over nonprime Galois fields do not satisfy Proposition 1, and are not suitable for IF equalization. Nevertheless, cyclic codes that are defined over $\mathbb{Z}_q$ are suitable for IF equalization even when q is not prime. An example of such a code, which is based on a binary cyclic code, is given in Section VII.

An IF equalizer (depicted in Fig. 2), rather than attempting to cancel most of the ISI (as in LE as well as DFE), ensures that the output of the FFE, which consists of both the desired signal and the ISI, can be decoded. Then, the transmitted codeword is recovered from the decoded noise-free signal. The closure property of cyclic codes w.r.t. integer-valued cyclic convolution modulo-q suggests an approach for achieving this objective. The transmitter sends a codeword from a cyclic code. The receiver uses the FFE for equalizing the channel's impulse response to an impulse response $I(D)$, such that all the coefficients of $I(D)$ are integers. As a result, the induced channel at the output of the FFE performs integer-valued convolution with the channel's input (ignoring the additive noise for the moment). This integer-valued convolution is made *cyclic* using a simple manipulation on the transmitted data. Reducing the FFE's output modulo-q results in cyclic convolution modulo-q of a codeword from a cyclic code and an integer-valued filter, which yields a codeword from the same cyclic code due to Proposition 1. This codeword can be decoded in order to eliminate the Gaussian noise. The original transmitted codeword can then be recovered using a DFE.

A. Related Work

IF equalization is related to lattice reduction which is a low-complexity detection scheme for MIMO channels where a PAM (or QAM) constellation is used for each one of the transmitted streams. Lattice reduction, originally introduced by Yao and Wornell [19] (see also [20]), is based on the observation that rather than directly detecting the PAM constellation points transmitted over each stream, one can first detect integer-valued linear combinations of these points and then recover the original points by solving a set of linear equations. For many channels this, essentially linear, approach significantly outperforms ZF-LE and successive interference cancellation (which is a common term for DFE over MIMO channels). IF equalization for MIMO channels, introduced by Zhan *et al.* [11], extends lattice reduction to coded transmission, where the PAM constellation is replaced by a single *linear* code which is used to encode each one of the streams. Accordingly, instead of decoding the transmitted codewords, the receiver aims to decode modulo-reduced integer-valued linear combinations of these codewords. The linearity of the codebook that is used for transmission of all streams ensures that these modulo-reduced linear combinations are codewords from the same linear codebook as well.

In [21], a connection between lattice-reduction and partial-response signaling for ISI channels (see, for example, [22]) was drawn. In particular, it was shown that equalizing the channel impulse response to an integer-valued one has several advantages, the most prominent of which is reducing the amount of CSI needed at the transmitter in order to perform precoding. The main contribution of our work is in combining cyclic codes with IF equalization for ISI channels. The improvement w.r.t. [21] is in showing that when cyclic codes are used, the DFE can be implemented at the receiver without suffering error propagation, and hence precoding is not necessary.

B. Detailed Description of IF Equalization

For simplicity of exposition, we limit ourselves in the sequel to real-valued channels (and transmission of real symbols). The extension of the proposed scheme to complex transmission is straightforward, and, unless stated otherwise, all results derived in this study hold for the complex case as well. Further, we only describe the ZF-IF approach which is suitable for the high SNR regime. As will be clear in the sequel, this is the regime where IF equalization is most effective. We now describe the operations taken by the transmitter and the receiver in our scheme.

Transmitter:

Encoding: The transmitter uses an $[N, K]$ linear cyclic codebook $\mathcal{C}$ which is defined over the ring $\mathbb{Z}_q$. Any linear code can be represented in a systematic form where the redundancy symbols appear at the beginning of the codeword, and the information symbols at the end of the codeword. We use such a representation for our encoder. The transmitter constructs each one of its message vectors using $K-(n-1)$ (the parameter n will be defined later) symbols from $\mathbb{Z}_q$ followed by $n-1$ zeros, resulting in message vectors of length K. Each such message vector is encoded, and due to the systematic manner of the encoder the resulting codeword ends with $(n-1)$ zeros. This zero-padding procedure serves several purposes, as will be explained in the sequel. The transmission rate is

$$R = \frac{K-(n-1)}{N} \cdot \log_2 q \text{ bits/channel use.}$$

We refer to the codeword produced by the encoder as $\mathbf{c}$

Meeting the Power Constraint: The encoder's output is an N-dimensional codeword, each element of which belongs to the constellation $\{0, 1, \ldots, q-1\} \in \mathbb{R}$. Since this constellation is not power-efficient, before transmitting the codeword $\mathbf{c}$ over the channel, the transmitter shifts and scales it to

$$\mathbf{x} = \sqrt{\frac{12\mathsf{SNR}}{q^2-1}}\left(\mathbf{c} - \frac{q-1}{2}\right).$$

For example, if $q = 2$ each element of the codeword $\mathbf{c}$ is mapped to $\pm\sqrt{\mathsf{SNR}}$. If the elements of the codeword $\mathbf{c}$ are uniformly distributed over $\mathbb{Z}_q$, this mapping ensures that[2]

$$\frac{1}{N}\mathbb{E}\left[\|\mathbf{x}\|^2\right] = \mathsf{SNR}.$$

Receiver:

Feed-Forward Equalization: The front-end of the receiver is an FFE

$$A_{\text{ZF-IF}}(D) = \frac{I(D)}{H(D)} \tag{3}$$

which equalizes the impulse response of the channel $H(D)$ to a causal integer-valued impulse response of length n (recall that n dictates the length of the zero padding)

$$I(D) = \sum_{k=0}^{n-1} i_k D^k.$$

We denote the vector of integer-valued coefficients of $I(D)$ by

$$\mathbf{i} = [i_0 \; i_1 \; \ldots \; i_{n-1}]^T.$$

The output of the FFE is

$$Y'(D) = X(D)I(D) + \tilde{Z}(D)$$

where $\tilde{Z}(D) = N(D)I(D)/H(D)$ is colored Gaussian noise with zero mean and variance

$$\sigma^2_{\tilde{Z}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|I(e^{jw})|^2}{|H(e^{jw})|^2} d\omega.$$

Note that taking $I(D) = 1$ is a special case in which the IF equalizer reduces to a ZF-LE one. In most cases, choosing $I(D)$ otherwise results in smaller noise enhancement. The criterion for determining the feed-forward filter for the IF equalizer will be given in Section IV.

Receiver:

Scaling, Adding Offset, and Modulo Reduction: We would like to bring the output of the FFE into the form

$$Y''(D) = [C(D)I(D) + Z(D)] \bmod q \tag{4}$$

where $C(D)$ is the D-transform of $\mathbf{c}$, and $Z(D)$ is the additive colored Gaussian noise. To that end, the receiver multiplies the FFE's output by

$$\sqrt{\frac{q^2-1}{12\mathsf{SNR}}}$$

and adds a factor of

$$\frac{q-1}{2}\mathbf{1}(D)I(D)$$

where we define $\mathbf{1}(D)$ as a causal sequence of ones

$$\mathbf{1}(D) = \sum_{k=0}^{N-1} D^k.$$

[2]Assuming $n \ll N$, the effect of the zero padding on the average transmission power is negligible.

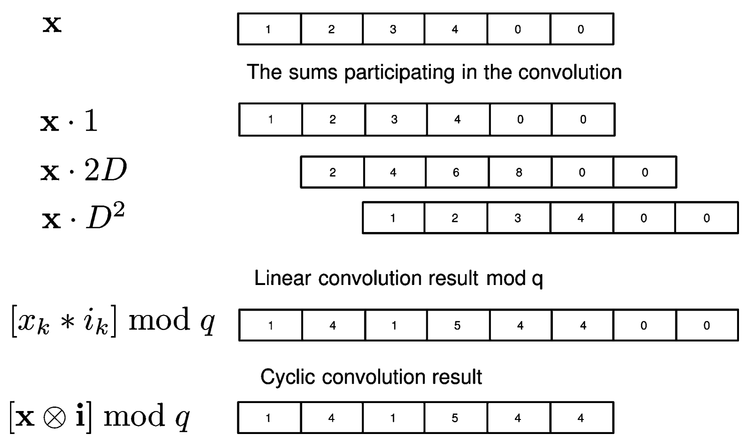

Fig. 3. Illustration of the equivalence between a linear convolution $x_k * i_k$ and a cyclic convolution $[\mathbf{x} \otimes \mathbf{i}] \bmod q$ when the last $n-1$ entries of $\mathbf{x}$ are zero. In this example, $I(D) = 1 + 2D + D^2$ and $q = 7$.

Finally, the obtained signal is reduced modulo-q which yields

$$\begin{aligned} Y''(D) &= \left[\sqrt{\frac{q^2-1}{12\mathsf{SNR}}}\left(X(D)I(D) + \tilde{Z}(D)\right)\right. \\ &\quad \left. + \frac{q-1}{2}\mathbf{1}(D)I(D)\right] \bmod q \\ &= \left[C(D)I(D) - \frac{q-1}{2}\mathbf{1}(D)I(D)\right. \\ &\quad \left. + \sqrt{\frac{q^2-1}{12\mathsf{SNR}}}\tilde{Z}(D) + \frac{q-1}{2}\mathbf{1}(D)I(D)\right] \bmod q \\ &= \Big[C(D)I(D) + Z(D)\Big] \bmod q \end{aligned}$$

where $Z(D) = \sqrt{(q^2-1)/12\mathsf{SNR}} \cdot \tilde{Z}(D)$.

The zero padding at the transmitter guarantees that the last $n-1$ entries of each codeword $\mathbf{c}$ are zeros. This has three desirable effects: first, as illustrated by Fig. 3, it ensures that the first N entries of the linear convolution modulo-q between $\mathbf{c}$ and $\mathbf{i}$ are equal to the cyclic convolution $[\mathbf{c} \otimes \mathbf{i}] \bmod q$, where with a slight abuse of notation when $\mathbf{i}$ participates in a cyclic convolution we assume it is padded by $N - n$ zeros such that its length is N. Second, the zero padding decouples different blocks of codewords by "clearing" the channel's memory and ensuring that there is no ISI between different blocks. Finally, the zero padding helps to recover the transmitted codeword as will be described in the sequel.

Let $\mathbf{y}'' = \left[y_0'' \; y_1'' \; \cdots \; y_{N-1}''\right]^T$ be a vector consisting of the first N samples of the equivalent channel's output. We have

$$\mathbf{y}'' = [\mathbf{c} \otimes \mathbf{i} + \mathbf{z}] \bmod q$$

where $\mathbf{z}$ consists of N consecutive samples of the process $Z(D)$. Due to the distributive law of the modulo operation, $\mathbf{y}''$ can be rewritten as

$$\begin{aligned} \mathbf{y}'' &= [[\mathbf{c} \otimes \mathbf{i}] \bmod q + \mathbf{z}] \bmod q \\ &= [\mathbf{c}' + \mathbf{z}] \bmod q \end{aligned} \tag{5}$$

where $\mathbf{c}' = [\mathbf{c} \otimes \mathbf{i}] \bmod q \in \mathcal{C}$ due to Proposition 1. Thus, $\mathbf{y}''$ is the output of an induced modulo-additive colored Gaussian noise channel with the codeword $\mathbf{c}'$ as input.

Decoding: In this step, $\mathbf{y}''$ enters a decoder that outputs an estimate $\hat{\mathbf{c}}'$ for the codeword $\mathbf{c}'$. We discuss the properties of the codebook $\mathcal{C}$ that ensure a small error probability in decoding $\mathbf{c}'$ in Section VI. In general, off-the-shelf cyclic codes over $\mathbb{Z}_q$ that perform well over an AWGN channel will perform well on the induced channel as well, in the high-SNR regime.

Recovering the Transmitted Codeword From the Decoded Codeword: After decoding $\mathbf{c}' = [\mathbf{c} \otimes \mathbf{i}] \bmod q$, we would like to recover the original codeword $\mathbf{c}$ from it. Note that $\mathbf{c}'$ can be thought of as the output of a noise-free modulo-q ISI channel, with input $\mathbf{c}$ and channel impulse response $[\mathbf{i}] \bmod q$. Therefore, it is possible to apply a modulo-q DFE for recovering $\mathbf{c}$ from $\mathbf{c}'$. Since the last $(n-1)$ symbols of $\mathbf{c}$ are known (i.e., they are all zeros), the DFE can be initialized with these values.

For simplicity of the DFE operation, it is convenient to restrict the filter $I(D)$ to be monic. This restriction does not imply a great loss of generality, as the optimal choice of $I(D)$ is in most cases monic anyway.

Remark 2: We note that recovering $\mathbf{c}'$ is possible even if $i_0 \neq 1$ as long as i_0 is invertible, i.e., if there exist an integer a such that $[a \cdot i_0] \bmod q = 1$. In this case, the receiver can first compute

$$\mathbf{c}'' = [a \cdot \mathbf{c}'] \bmod q = [\mathbf{c} \otimes (a \cdot \mathbf{i})] \bmod q$$

and then apply the DFE with the monic filter $[a \cdot \mathbf{i}] \bmod q$ on $\mathbf{c}''$. In fact, if q is a prime number, one can recover $\mathbf{c}'$ for any choice of $\mathbf{i}$, since each entry of $[\mathbf{i}] \bmod q$ is either zero or invertible, and at least one of the entries is invertible.[3] Assume the first nonzero entry of $[\mathbf{i}] \bmod q$ is i_k. Then, the receiver can cyclicly shift $\mathbf{c}'$ by k places, and then apply a DFE with the filter $\tilde{\mathbf{i}} = [i_k \; \cdots \; i_{n-1}]$ on the cyclicly shifted $\mathbf{c}'$.

The complete IF equalization system is depicted in Fig. 4.

We end this section by remarking that the recently proposed "signal codes" [23], though not cyclic, are also suitable for IF equalization. Signal codes are a special family of lattice codes which are linear shift-invariant over the reals. In [23], it is demonstrated that these codes can operate reasonably close to

[3]Otherwise, all entries are multiples of q, and better performance can be obtained using the integer-valued filter $I(D)/q$.

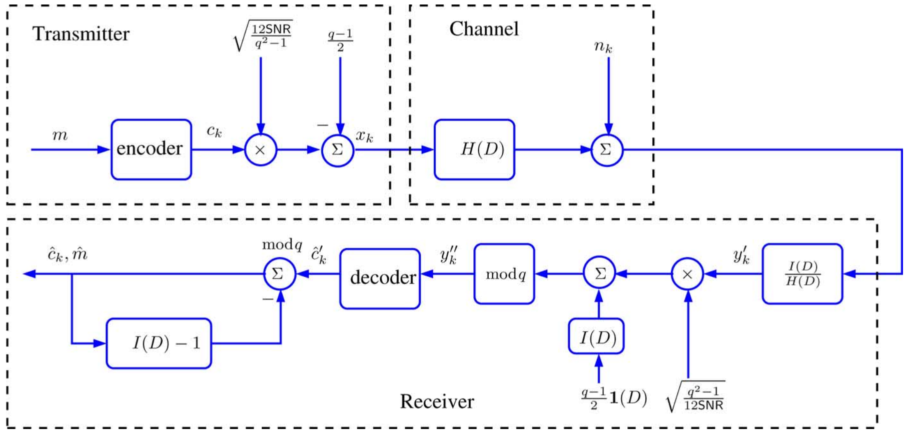

Fig. 4. Detailed description of an IF-DFE system (assuming $i_0 = 1$).

capacity with an acceptable complexity. Therefore, this class of codes may be attractive for IF equalization.

IV. Criterion for Choosing the Target Integer Channel Filter

We wish to find an integer-valued filter $I(D)$ of length smaller than (or equal to) n such that the noise enhancement experienced by the ZF-IF equalizer for a given channel $H(D)$ is minimized.[4] We denote the variance of the noise process $\tilde{Z}(D)$ at the output of the IF front-end filter (3) by $\sigma^2_{\text{ZF-IF-DFE}}(\mathbf{i})$, which is given by

$$\sigma^2_{\text{ZF-IF-DFE}}(\mathbf{i}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|I(e^{j\omega})|^2}{|H(e^{j\omega})|^2}d\omega. \quad (6)$$

Denote

$$K(D) = \frac{1}{H(D)H^*(D^{-*})}. \quad (7)$$

Thus, k_j is an autocorrelation sequence (and in particular $k_{-j} = k_j$ as we assume $H(D)$ to be real).

By straightforward algebra, (6) may be written as a quadratic form

$$\sigma^2_{\text{ZF-IF-DFE}}(\mathbf{i}) = \mathbf{i}^T\tilde{\text{K}}_n\mathbf{i}$$

where $\tilde{\text{K}}_n$ is the positive semidefinite Toeplitz matrix

$$\tilde{\text{K}}_n = \begin{bmatrix} k_0 & k_{-1} & k_{-2} & \dots & k_{-(n-1)} \\ k_1 & k_0 & k_{-1} & \dots & k_{-(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n-1} & k_{n-2} & k_{n-3} & \dots & k_0 \end{bmatrix}.$$

Let F_n be any matrix satisfying

$$\tilde{\mathbf{K}}_n = \mathbf{F}_n{}^T\mathbf{F}_n.$$

We therefore have

$$\sigma^2_{\text{ZF-IF-DFE}}(\mathbf{i}) = \mathbf{i}^T\tilde{\mathbf{K}}_n\mathbf{i} = \mathbf{i}^T\mathbf{F}_n{}^T\mathbf{F}_n\mathbf{i} = ||\mathbf{F}_n\mathbf{i}||^2. \quad (8)$$

Equation (8) implies that finding the optimal (ZF) integer-valued filter $I(D)$ of length smaller than (or equal to) n is equivalent to finding the shortest vector in the lattice $\Lambda(\text{F}_n)$, which is composed of all integral combinations of the columns of F_n, i.e.,

$$\Lambda(\text{F}_n) = \{\lambda = \text{F}_n\mathbf{a} : \mathbf{a} \in \mathbb{Z}^n\}. \quad (9)$$

We note that in [21] a criterion for finding the optimal ZF-IF monic filter $I(D)$ was established. Although the formulation of the criterion of [21] is different from that presented here, it can be verified that the two criteria are identical. This follows by noting that the optimization problem in [21] amounts to finding the shortest vector in a lattice spanned by a matrix G_n that satisfies $\tilde{\text{K}}_n = \text{G}_n^T\text{G}_n$.

Finding the shortest lattice vector is known to be NP hard, but fortunately efficient suboptimal algorithms for finding a short lattice basis are known. An important representative of this class of algorithms is the celebrated LLL algorithm [24], which has polynomial complexity and usually gives adequate results in practice. In order to find a "good" integer-valued filter $I(D)$, we may therefore apply the LLL algorithm on F_n. The algorithm's result is a new basis (of short vectors) for F_n. We then need to find the shortest vector $\mathbf{v}$ in this basis and choose $\mathbf{i} = \text{F}_n{}^{-1}\mathbf{v}$. In the next section, we derive an upper bound on the induced noise enhancement when the true (optimal) shortest vector of the lattice is used, which serves as a useful benchmark.

V. Upper Bounds on the Noise Enhancement of Optimal ZF-IF Equalization

We now upper bound the noise enhancement induced by optimal ZF-IF equalization. Throughout this section, we assume that the channel has a finite-length impulse response of length $p+1$, i.e.,

$$H(D) = \sum_{k=0}^{p} h_k D^k.$$

Definition 2: We define the noise variance at the output of the FFE (3) when the optimal filter $I(D)$ with length smaller than (or equal to) n is used, by

$$\sigma^{2*}_{\text{ZF-IF-DFE}}(n) = \min_{\mathbf{i}\in\mathbb{Z}^n}\sigma^2_{\text{ZF-IF-DFE}}(\mathbf{i}).$$

[4]In fact, we would like to minimize the variance of $Z(D)$, but since it is just a scaled version of the process $\tilde{Z}(D)$, the two problems are equivalent.

For an optimal filter $I(D)$ of arbitrary length, the noise variance is therefore

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}} = \min_{m>0} \sigma^{2*}_{\mathrm{ZF-IF-DFE}}(m).$$

We proceed to upper bound $\sigma^{2*}_{\mathrm{ZF-IF-DFE}}(n)$ and $\sigma^{2*}_{\mathrm{ZF-IF-DFE}}$ using known results from the theory of lattices, and the theory of Toeplitz matrices.

Definition 3: We define $\lambda_1(\Lambda(\mathrm{F}_m))$ to be the length of the shortest vector in the lattice $\Lambda(\mathrm{F}_m)$, as was defined in Section IV. Namely

$$\lambda_1(\Lambda(\mathrm{F}_m)) = \min_{\mathbf{a}\in\mathbb{Z}^m} \|\mathrm{F}_m\mathbf{a}\|.$$

Lemma 1: (Minkowski) If F_m is of rank m, then

$$\lambda_1(\Lambda(\mathrm{F}_m)) \le \left(\frac{2^m}{\beta_m}\right)^{\frac{1}{m}} |\det(\mathrm{F}_m)|^{\frac{1}{m}}$$

where β_m is the constant of proportionality of the volume of an m-dimensional ball, i.e., the volume of an m-dimensional ball with radius R is given by $V_m(R) = \beta_m R^m$.

Proof: See, e.g., [25]. ■

The constant β_m can be bounded by (see, e.g., [26])

$$\beta_m > \left(\frac{2\pi e}{m}\right)^{m/2} \frac{1}{\sqrt{1.4\pi m}}.$$

Recalling that $\tilde{\mathrm{K}}_m = {\mathrm{F}_m}^T\mathrm{F}_m$, we thus have

$$\lambda_1^2(\Lambda(\mathrm{F}_m)) < \eta(m)\cdot\left[\det(\tilde{\mathrm{K}}_m)\right]^{\frac{1}{m}} \tag{10}$$

where

$$\eta(m) = \frac{2m}{\pi e}\cdot(1.4\pi m)^{\frac{1}{m}}. \tag{11}$$

In Section IV, it was shown that

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}}(n) = \lambda_1^2(\Lambda(\mathrm{F}_n)).$$

Therefore, for any value of n we have

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}}(n) < \eta(n)\cdot\left[\det(\tilde{\mathrm{K}}_n)\right]^{\frac{1}{n}}. \tag{12}$$

Obviously, allowing for a longer filter enlarges the optimization space and can only result in smaller noise enhancement. Thus, $\sigma^{2*}_{\mathrm{ZF-IF-DFE}}(n)$ is monotonically nonincreasing in n. Nevertheless, the r.h.s. of (12) is not monotone in n. For this reason, a tighter bound is obtained by taking the minimum of the r.h.s. over all possible values of $m \le n$

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}}(n) < \min_{n\ge m>0}\left[\eta(m)\cdot\left[\det(\tilde{\mathrm{K}}_m)\right]^{\frac{1}{m}}\right]. \tag{13}$$

Allowing for arbitrary n in (13) gives

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}} < \min_{m>0}\left[\eta(m)\cdot\left[\det(\tilde{\mathrm{K}}_m)\right]^{\frac{1}{m}}\right]. \tag{14}$$

The bounds (13) and (14) give little insight about the loss of IF equalization w.r.t. the optimal performance. Fortunately, using results from the theory of Toeplitz matrices, it is possible to give a closed-form upper bound on $\det\left(\tilde{\mathrm{K}}_m\right)$. As a consequence, insightful expressions can be obtained. To that end, define

$$\alpha_H = \frac{|z_0 z_1 \dots z_{p-1}|^{2p}}{\prod_{\mu,\nu}|z^*_\mu z_\nu - 1|} \tag{15}$$

where $z_0, z_1, \dots, z_{p-1}$ are the maximum-phase zeros of $H(D)H^*(D^{-*})$ (i.e., the zeros outside the unit circle).

The next result is due to Grenander and Szegö, and the proof of which can be found in [27, Chapter 5].

Lemma 2: Let $H(D) = \sum_{k=0}^{p} h_k D^k$ be a polynomial of degree p. Further, let $K(D)$ be as defined in (7). Then for every $m \ge p+1$

$$\left[\det(\tilde{\mathrm{K}}_m)\right]^{\frac{1}{m}} = (\alpha_H)^{\frac{1}{m}}\cdot\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log K(e^{j\omega})d\omega\right]$$

where α_H is defined in (15).

Lemma 2 may be extended to an inequality which holds for any $m > 0$, as stated by the following corollary.

Corollary 1: Let $H(D) = \sum_{k=0}^{p} h_k D^k$ be a polynomial of degree p. Further, let $K(D)$ be as defined in (7). Then for every $m > 0$

$$\left[\det(\tilde{\mathrm{K}}_m)\right]^{\frac{1}{m}} \le (\alpha_H)^{\frac{1}{m}}\cdot\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log K(e^{j\omega})d\omega\right] \tag{16}$$

where α_H is defined in (15).

Proof: For the case $m \ge p+1$, inequality (16) follows directly from Lemma 2. Therefore, it suffices to consider the case $m < p+1$. Define the (stationary) noise process $Z^{\mathrm{ZF}}(D) = N(D)/H(D)$ and note that k_j is the autocorrelation function of this process, and $\tilde{\mathrm{K}}_m$ is the autocorrelation matrix of m consecutive samples of this process. Let σ^2_ℓ be the minimum mean squared one-step prediction error of the kth sample of the noise process, z^{ZF}_k, from the ℓ samples $\{z^{\mathrm{ZF}}_{k-1},\dots,z^{\mathrm{ZF}}_{k-\ell}\}$. For all $\ell \ge 0$, we have (see, e.g., [2])

$$\sigma^2_\ell \ge \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log K(e^{j\omega})d\omega\right]. \tag{17}$$

It is well known [28] that

$$\sigma^2_\ell = \frac{\det(\tilde{\mathrm{K}}_{\ell+1})}{\det(\tilde{\mathrm{K}}_\ell)}. \tag{18}$$

Thus, for $m < p+1$

$$\begin{aligned}\frac{\det(\tilde{\mathrm{K}}_{p+1})}{\det(\tilde{\mathrm{K}}_m)} &= \prod_{\ell=m}^{p}\sigma^2_\ell \\ &\ge \left(\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log K(e^{j\omega})d\omega\right]\right)^{p+1-m}.\end{aligned} \tag{19}$$

Substituting the exact expression for $\det(\tilde{\mathrm{K}}_{p+1})$ which is given by Lemma 2 establishes the desired result. ■

We are now ready to present Theorem 1 which is the main result of this section.

Theorem 1: Assume that the channel has a finite-length impulse response $H(D) = \sum_{k=0}^{p} h_k D^k$. For an optimal choice of the integer filter $I(D)$ (of arbitrary length), the noise power at the output of the filter $I(D)/H(D)$ is bounded by[5]

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}} \leq \sigma^2_{\mathrm{ZF-DFE}} \cdot \min_{m>0} \left[\eta(m) \cdot (\alpha_H)^{\frac{1}{m}}\right] \tag{20}$$

where $\eta(m)$ is defined in (11), α_H is defined in (15), and

$$\sigma^2_{\mathrm{ZF-DFE}} = \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|H(e^{j\omega})|^2 d\omega\right]. \tag{21}$$

Proof: Corollary 1 may be rewritten as

$$\left[\det(\tilde{\mathrm{K}}_m)\right]^{\frac{1}{m}} \leq (\alpha_H)^{\frac{1}{m}} \cdot \sigma^2_{\mathrm{ZF-DFE}} \tag{22}$$

where $\sigma^2_{\mathrm{ZF-DFE}}$, given by (21), is the variance of the AWGN after the optimal FFE is applied in ZF-DFE (see, e.g., [7]). Substituting (22) into (13) yields

$$\sigma^{2*}_{\mathrm{ZF-IF-DFE}}(n) < \sigma^2_{\mathrm{ZF-DFE}} \cdot \min_{n\geq m>0} \left[\eta(m) \cdot (\alpha_H)^{\frac{1}{m}}\right]. \tag{23}$$

Allowing for arbitrary n in (23) results in (20) which proves the theorem. ■

The additional noise enhancement caused by the ZF-IF equalizer w.r.t. an optimal ZF-DFE as bounded in (20) consists of two factors: $(\alpha_H)^{\frac{1}{m}}$ and $\eta(m)$. The factor $(\alpha_H)^{\frac{1}{m}}$ is greater than (or equal to) 1 and tends to 1 as $m \to \infty$. On the other hand, the factor $\eta(m)$ is (approximately) linearly increasing with m. The minimization in (20) strikes a balance (i.e., searches for the optimal tradeoff) between these two factors.

The tightness of the bound depends on the tightness of the Minkowski bound of Lemma 1. It is known that there exist lattices of small dimensions for which this bound is rather tight. Nevertheless, for "most" lattices the shortest lattice vector is much shorter than what this bound predicts. In fact, the tightness of the Minkowski bound for a certain lattice is related to the "goodness" of that lattice for sphere packing [26]. We further note that the family of lattices considered in this paper is not general, as $\mathrm{F}_m^T\mathrm{F}_m$ is a Toeplitz matrix, and therefore the Minkowski bound may always be loose, as we further discuss in Section VI.

VI. Performance of ZF-IF Equalization

The capacity of the Gaussian ISI channel (1) at high SNR is given by (see, e.g., [7])

$$C = \frac{1}{2}\log_2\left(\frac{\mathsf{SNR}}{\sigma^2_{\mathrm{ZF-DFE}}}(1+o(1))\right) \tag{24}$$

where $\sigma^2_{\mathrm{ZF-DFE}}$ was defined in (21), and where $o(1) \to 0$ as $\mathsf{SNR} \to \infty$. Note that (24) is valid only for channels for which $\sigma^2_{\mathrm{ZF-DFE}}$ is finite, i.e., for channels satisfying the Paley–Wiener condition (see [2]).

In this section, we analyze the total gap-to-capacity of the ZF-IF equalization scheme at high SNR, i.e., the gap between the performance obtained using ZF-IF and the optimal performance (24).

The ZF-IF scheme, as described in Section III, transforms the original Gaussian ISI channel into the induced modulo-additive colored Gaussian noise channel (5). For the analysis in this section, it is convenient to scale $\mathbf{y}''$ by a factor of $1/q$, resulting in

$$\tilde{\mathbf{y}} = \frac{1}{q}\mathbf{y}'' = \left[\frac{1}{q}\mathbf{c}' + \mathbf{z}'\right] \mathrm{mod}1 \tag{25}$$

where $\mathbf{z}' = \mathbf{z}/q$ consists of N consecutive samples of a colored Gaussian noise process with variance

$$\sigma^2_{Z'} = \frac{\sigma^2_Z}{q^2} = \frac{q^2-1}{q^2} \cdot \frac{\sigma^2_{\mathrm{ZF-IF-DFE}}(\mathbf{i})}{12\mathsf{SNR}}. \tag{26}$$

As we recall, correct decoding of $\mathbf{c}'$ ensures correct reconstruction of $\mathbf{c}$.

In order to analyze the performance limits of the induced channel, we lower bound the average mutual information

$$\frac{1}{N}I\left(\mathbf{C}';\tilde{\mathbf{Y}}\right)$$

between its input and output. The average mutual information corresponding to a certain distribution on the channel's input gives the highest possible rate for reliable communication, when a random channel code which is drawn according to that distribution is used [29]. However, the input to the induced channel (25) is confined to be a cyclic code (as opposed to a random code), and thus the average mutual information may never be achieved. Nevertheless, the average mutual information is a useful metric for the performance of ZF-IF if we account for the possible loss of rate due to the use of a cyclic code.

The channel (25) is a modulo-additive *colored* Gaussian noise channel, where the variance of the noise is $\sigma^2_{Z'}$. The mutual information between the input and output of such a channel is maximized by a uniform memoryless distribution over the modulo interval. If $\mathbf{C}$ has a memoryless uniform distribution over $\mathbb{Z}_q$, then $\mathbf{C}/q$ has a memoryless uniform distribution over the set $[0 \; 1/q \; \cdots \; (q-1)/q] \in \mathbb{R}$. For large q, $\mathbf{C}/q$ approaches a memoryless uniform distribution over the interval $[0,1)$, and consequently, the distribution of $\tilde{\mathbf{Y}}$ approaches a uniform memoryless distribution over the same interval. Let Z_{AWGN} be a Gaussian random variable with zero mean and variance $\sigma^2_{Z'}$. In the limit of $q \to \infty$, we therefore have

$$\begin{aligned}\frac{1}{N}I\left(\mathbf{C}';\tilde{\mathbf{Y}}\right) &= \frac{1}{N}\left(h(\tilde{\mathbf{Y}}) - h(\mathbf{Z}'\mathrm{mod}1)\right)\\ &\geq \frac{1}{N}\left(N\cdot\log_2(1) - h(\mathbf{Z}')\right) \qquad (27)\\ &\geq -h(Z_{\mathrm{AWGN}}) \qquad (28)\\ &= -\frac{1}{2}\log_2(2\pi e\cdot\sigma^2_{Z'})\\ &> \frac{1}{2}\log_2\left(\frac{\mathsf{SNR}}{\sigma^2_{\mathrm{ZF-IF-DFE}}(\mathbf{i})}\cdot\frac{12}{2\pi e}\right)\end{aligned}$$

where (27) follows since $\mathbf{C}'$, and hence $\tilde{\mathbf{Y}}$, is i.i.d. uniformly distributed over $[0,1)$ and since the modulo operation can only decrease differential entropy. Inequality (28) follows from the fact that the average differential entropy of a colored Gaussian

[5]For (complex) transmission over a complex channel, (11) changes to

$$\eta(m) = \frac{4m}{\pi e}\cdot(2.8\pi m)^{\frac{1}{2m}}.$$

noise is always smaller than that of a white Gaussian noise with the same variance. Let us express $\sigma^2_{\mathrm{ZF-IF-DFE}}(\mathbf{i})$ as a product of two factors

$$\sigma^2_{\mathrm{ZF-IF-DFE}}(\mathbf{i}) = \gamma_{I,H} \cdot \sigma^2_{\mathrm{ZF-DFE}}$$

where $\gamma_{I,H}$ is the additional noise enhancement caused by the FFE in ZF-IF with the filter $I(D)$ w.r.t. that caused by the FFE of ZF-DFE. With this notation, we have

$$\frac{1}{N} I\left(\mathbf{C}'; \tilde{\mathbf{Y}}\right) > \frac{1}{2}\log_2\left(\frac{\mathsf{SNR}}{\sigma^2_{\mathrm{ZF-DFE}}} \cdot \frac{1}{\gamma_{I,H}} \cdot \frac{12}{2\pi e}\right) = \frac{1}{2}\log_2\left(\frac{\mathsf{SNR}/\Gamma_{I,H}}{\sigma^2_{\mathrm{ZF-DFE}}}\right)$$

where

$$\Gamma_{I,H} = \frac{2\pi e}{12} \cdot \gamma_{I,H}$$

is the SNR loss due to IF-ZF equalization. It follows that the gap-to-capacity in dB (at high SNR) is given by[6]

$$10\log_{10}(\Gamma_{I,H}) = 10\log_{10}\left(\frac{2\pi e}{12}\right) + 10\log_{10}(\gamma_{I,H}). \quad (29)$$

The first term on the r.h.s. of (29) is the well-known high-SNR shaping gain, which equals 1.53 dB. The reason for this loss is the uniform distribution of the output of the induced channel $Y''(D)$. Due to the 1-D modulo operation, the output of the induced channel is uniform at all SNRs. Note that at high SNR the modulo operation incurs no loss since the output is uniformly distributed as a result of the input distribution being uniform. The modulo operation allows the decoder to use the original codebook $\mathcal{C}$ for decoding $\mathbf{c}'$ which preserves the original decoding complexity of the codebook. However, decoding the induced channel's output after the modulo reduction is equivalent to searching for the point that was most likely transmitted over the *infinite* lattice $\mathcal{C} + q\mathbb{Z}^N$. This is strictly suboptimal since no more than 2^{NR} points of the infinite lattice, which correspond to $\mathcal{C} \otimes \mathbf{i}$, are valid, and hence better performance can be achieved by searching only over these valid points at the decoder.

While the loss for a uniform output amounts to 1.53 dB at high SNR, at low SNR the loss may be significantly greater (see, e.g., [30]), which makes IF equalization less attractive in that regime. In the low-SNR regime, this loss can be mitigated by avoiding the modulo reduction at the receiver. Implementing such modifications at the receiver without significantly increasing the computational complexity is an interesting avenue for future research.

The second loss in (29) is related to the additional noise enhancement caused by the FFE. We denote the additional noise enhancement when the optimal integer-valued filter (of arbitrary length) is used by

$$\gamma_H = \min_{I(D)} \gamma_{I,H}.$$

It follows from Theorem 1 that γ_H is upper bounded by

$$\gamma_H \le \min_{m>0}\left[\eta(m) \cdot (\alpha_H)^{\frac{1}{m}}\right]. \quad (30)$$

[6]This is the gap-to-capacity in the case of (complex) transmission over complex channels as well.

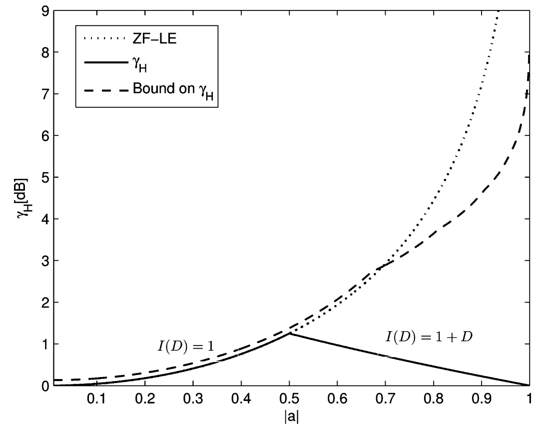

Fig. 5. Gap γ_H for the real two-tap channel $H(D) = 1 + aD^p$. The solid line shows γ_H for *any* value of p, the dashed line shows the bound (30) for $p = 1$, and the dotted line shows the noise enhancement induced by ZF-LE.

In order to gain more insight into the term $\gamma_{I,H}$, we illustrate the effect of the noise enhancement through the following examples of ISI channels.

Example 1—Two-Tap Real "RAKE" Channel: An interesting example is a *real* channel with only two nonzero taps (which may be spaced arbitrarily far apart), namely $H(D) = 1 + aD^p$. Without loss of generality, we can assume that $|a| \le 1$, as otherwise we can transform the channel into the channel $\tilde{H}(D) = 1 + \frac{1}{a}D^p$ by using an all-pass filter. It is rather obvious that the optimal choice for $I(D)$ is either $I_0(D) = 1$ or $I_1(D) = 1 + \frac{a}{|a|}D^p$. It can be shown by straightforward algebra that for the choice $I_0(D)$, the noise enhancement is given by $1/(1-a^2)$, and for the choice $I_1(D)$ the noise enhancement is given by $2/(1+|a|)$. It follows that for $|a| \le 1/2$, $I_0(D)$ is better, while for $1/2 < |a| \le 1$, $I_1(D)$ is better, and the maximum noise enhancement (which occurs for $|a| = 1/2$) is $4/3 \approx 1.25$ dB. Since for this channel $\sigma^2_{\mathrm{ZF-DFE}} = 1$, the total noise enhancement amounts to $\gamma_{I,H}$. Fig. 5 depicts γ_H for the real two-tap channel, along with the bound (30) evaluated for the same channel, and the noise enhancement caused by a ZF-LE. It is evident from the figure that for this channel the bound is not tight.

Example 2—Two-Tap Complex "RAKE" Channel: Consider the channel from the previous example $H(D) = 1 + aD^p$, where now $a = re^{j\theta}$ is a complex number. As before, we assume without loss of generality that $|a| \le 1$. As $H(D)$ is now a complex channel, we let $I(D)$ take complex integer values (Gaussian integers). In contrast to the real-valued two-tap channel, in this case there is no clear choice of $I(D)$ for each value of a. Nevertheless, similar to the real two-tap channel, γ_H is independent of the delay p between the first and the second tap (as is also the case for the ZF-LE). This follows since the noise enhancement caused by each of the filters $A_1(D) = I(D)/H(D)$ and $A_2(D) = I(D^p)/H(D^p)$ is the same. Therefore, if for the channel $H(D) = 1 + aD$, a certain choice of $I(D)$ results in noise enhancement of $\gamma_{I,H}$, then for the channel $\tilde{H}(D) = 1 + aD^p$, the choice $\tilde{I}(D) = I(D^p)$ results in $\gamma_{I,H}$ as well. While the performance of a ZF-LE is independent of the phase θ, the noise enhancement caused by the ZF-IF equalizer is significantly influenced by θ. It is

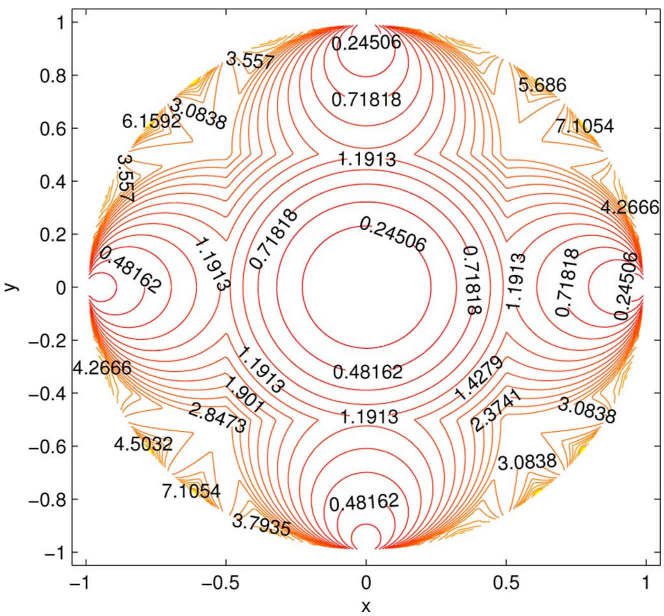

Fig. 6. Gap $\gamma_{I,H}$ (in dB) for the complex two-tap channel $H(D) = 1 + (x + j \cdot y)D^p$, when the optimal integer-valued filter $I(D)$ of length smaller than (or equal to) $5(p+1)$ is used.

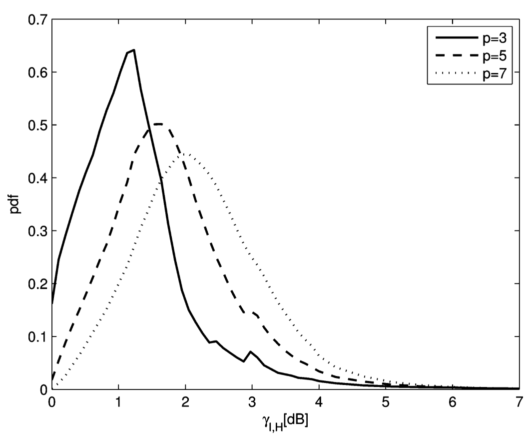

Fig. 7. Probability density function (pdf) of $\gamma_{I,H}$ (in dB) for the ISI channel $H(D) = \sum_{k=0}^{p} h_k D^p$ where $\{h_k\}_{k=0}^{p}$ are i.i.d. random Gaussian variables with zero mean and unit variance, when the optimal integer-valued filter $I(D)$ of length smaller than (or equal to) $5(p+1)$ is used.

interesting to note, however, that for the two-tap channel, the bound (30) (in its complex form) is independent of θ. Fig. 6 depicts $\gamma_{I,H}$ (in dB) for the two-tap channel with an optimal choice of $I(D)$ (which was found numerically by searching over all filters with Gaussian integer coefficients of length $n \leq 5(p+1)$), for values of $|a| \leq 0.99$.

Example 3—An ISI Channel With Random Taps: In this example, we consider the channel

$$H(D) = \sum_{k=0}^{p} h_k D^p$$

where $\{h_k\}_{k=0}^{p}$ are i.i.d. real random Gaussian variables with zero mean and unit variance. We numerically evaluate the probability density function (pdf) of $\gamma_{I,H}$ in dB for the optimal choice of $I(D)$ over all integer-valued filters of length $n \leq 5(p+1)$. Fig. 7 depicts the results for $p = 3$, $p = 5$, and $p = 7$. The results show that γ_H becomes larger when the channel is longer.

The third loss incurred by ZF-IF equalization is related to the constraint that the channel code is cyclic. The gap-to-capacity of ZF-IF equalization at a certain block error probability is the sum of the shaping loss (in dB), the additional noise enhancement $\gamma_{I,H}$ (in dB), and the gap-to-capacity (in dB, for the modulo Gaussian channel) of the chosen cyclic code at the desired block error probability. To the best of the authors' knowledge, it is still an open question whether cyclic codes can attain the capacity of the AWGN channel, let alone, that of the modulo Gaussian channel. Nevertheless, algebraic cyclic codes over $\mathbb{Z}_2$ (such as BCH for example) have been extensively used in communication systems for decades due to the fair tradeoff they offer between performance and complexity. Modern (e.g., LDPC) cyclic codes are a subject of extensive recent research, and some families of such codes were reported to have very good performance over the AWGN channel; see e.g., [31] and [32].

A subtle issue that deserves attention is the statistics of the additive noise after the FFE in ZF-IF equalization. While a proper choice of $I(D)$ usually allows us to substantially reduce the memory of this noise, its power spectrum density (PSD) is not strictly flat, i.e., it is colored. Since the proposed scheme crucially relies on the linear time-invariant structure of the code and the channel, it is not clear how to incorporate an interleaver. Therefore, it is desirable that the code that is used is not sensitive to the memory of the noise. For additive noise channels with unit noise variance, random Gaussian codebooks with nearest neighbor decoding are known to achieve any rate below $1/2 \log_2(1 + \mathsf{SNR})$ regardless of the exact noise statistics [33]. Thus, in general there is no problem with using AWGN codebooks for a colored Gaussian noise channel (assuming one is targeting toward the capacity of an AWGN with variance equal to that of the colored noise). However, our scheme is restricted to using cyclic codes, and hence exploring the robustness of this family of codes to memory in the noise is an interesting question for future research. As a positive example, note that cyclic algebraic codes, such as BCH codes, with hard decoding are only affected by the number of errors within the block, and are therefore robust to the noise memory.

Most of the research effort on linear cyclic codes has been devoted to binary codes. In the high-SNR regime, which is the focus of this paper, the cardinality of the code alphabet must be greater than 2, and a binary code does not suffice. In the next section, we propose a practical coded modulation scheme for high transmission rates that utilizes binary cyclic codes and is suitable for cyclic-coded IF equalization.

VII. Practical Coding at High Rates

In order to achieve high transmission rates, using a binary channel code, we propose a simple construction of a q-ary cyclic codebook which is obtained from a combination of a single coded layer and uncoded layers. This construction, which results in an Ungerboeck set partitioning-type coded modulation scheme (see [1]), utilizes coded and uncoded layers over $\mathbb{Z}_2$ in order to obtain a codebook over $\mathbb{Z}_{2^L}$, while essentially maintaining the encoding and decoding complexity of a single binary code. It can also be viewed as a variant of Construction A for lattices [26].

Let $\mathcal{C}_b$ be a binary cyclic code. The 2^L-ary codebook $\mathcal{C}$ is defined by

$$\mathcal{C} = \Big\{ \mathbf{c} \in \mathbb{Z}_{2^L}^N : \mathbf{c} = \mathbf{c}_b + \sum_{l=1}^{L-1} 2^l \mathbf{c}_{u,l}, \quad \mathbf{c}_b \in \mathcal{C}_b,\ \mathbf{c}_{u,l} \in \mathbb{Z}_2^N \text{ for } l = 1, \dots, L-1 \Big\}. \quad (31)$$

The codebook $\mathcal{C}$ is nothing more than a mapping of a binary coded layer $\mathbf{c}_b$ and $L-1$ binary uncoded layers $\{\mathbf{c}_{u,l}\}_{l=1}^{L-1}$ to $\mathbb{Z}_{2^L}^N$, in a manner that replicates the linearity and cyclic properties of the codebook $\mathcal{C}_b$ in $\mathbb{Z}_2$ to the larger ring $\mathbb{Z}_{2^L}$, as stated by the next lemma.

Lemma 3: The codebook $\mathcal{C}$ which is constructed according to (31) is a linear cyclic code of length N over the ring $\mathbb{Z}_{2^L}$.

Proof: Let $\mathbf{c}^1$ and $\mathbf{c}^2$ be two members of the codebook $\mathcal{C}$. Since any vector in $\mathbb{Z}_{2^L}$ can be decomposed into a sum of binary vectors multiplied by powers of 2, we can write

$$\mathbf{c}^1 = \mathbf{c}_b^1 + \sum_{l=1}^{L-1} 2^l \mathbf{c}_{u,l}^1$$

and

$$\mathbf{c}^2 = \mathbf{c}_b^2 + \sum_{l=1}^{L-1} 2^l \mathbf{c}_{u,l}^2.$$

It follows from the definition of the codebook $\mathcal{C}$ that $\mathbf{c}_b^1 \in \mathcal{C}_b$ and $\mathbf{c}_b^2 \in \mathcal{C}_b$. In order to show that $\mathcal{C}$ is linear over $\mathbb{Z}_{2^L}$, we have to show that

$$\left[\mathbf{c}^1 + \mathbf{c}^2\right] \bmod 2^L \in \mathcal{C}.$$

The modulo-2^L sum can be decomposed as

$$\left[\mathbf{c}^1 + \mathbf{c}^2\right] \bmod 2^L = \mathbf{w}_0 + \sum_{l=1}^{L-1} 2^l \mathbf{w}_l$$

for some binary vectors $\mathbf{w}_0, \dots, \mathbf{w}_{L-1} \in \mathbb{Z}_2^N$. It follows from the definition of $\mathcal{C}$ that $\left[\mathbf{c}^1 + \mathbf{c}^2\right] \bmod 2^L \in \mathcal{C}$ iff $\mathbf{w}_0 \in \mathcal{C}_b$. On the other hand

$$\begin{aligned} \mathbf{w}_0 &= \left[\left[\mathbf{c}^1 + \mathbf{c}^2\right] \bmod 2^L\right] \bmod 2 \\ &= \left[\mathbf{c}^1 + \mathbf{c}^2\right] \bmod 2 \qquad (32) \\ &= \left[\mathbf{c}_b^1 + \mathbf{c}_b^2\right] \bmod 2 \in \mathcal{C}_b \qquad (33) \end{aligned}$$

where (32) follows from the properties of the modulo operation, and (33) from the linearity of $\mathcal{C}_b$. This proves that $\mathcal{C}$ is linear over $\mathbb{Z}_{2^L}$. It is left to show that $\mathcal{C}$ is cyclic. Denote a one-position cyclic shift of a vector $\mathbf{c}$ by $\vec{\mathbf{c}}$. We would like to show that $\vec{\mathbf{c}}^1 \in \mathcal{C}$. We have

$$\vec{\mathbf{c}}^1 = \vec{\mathbf{c}}_b^1 + \sum_{l=1}^{L-1} 2^l \vec{\mathbf{c}}_{u,l}^1.$$

Since $\mathcal{C}_b$ is cyclic, $\vec{\mathbf{c}}_b^1 \in \mathcal{C}_b$, and hence $\vec{\mathbf{c}}^1 \in \mathcal{C}$. ■

The codebook $\mathcal{C}$ is therefore suitable for IF equalization. It is important to note that with this construction the last $n-1$ bits of the uncoded layers, as well as those of the coded layer, have to be zero in order to get the desired zero-padding effect.

When $\mathcal{C}$ is used with IF equalization, the resulting induced channel is (see Section III)

$$\mathbf{y}'' = [\mathbf{c}' + \mathbf{z}] \bmod 2^L \qquad (34)$$

where

$$\mathbf{c}' = \mathbf{c}_b' + \sum_{l=1}^{L-1} 2^l \mathbf{c}_{u,l}'. \qquad (35)$$

Substituting (35) into (34) gives

$$\mathbf{y}'' = \left[\mathbf{c}_b' + \sum_{l=1}^{L-1} 2^l \mathbf{c}_{u,l}' + \mathbf{z}\right] \bmod 2^L. \qquad (36)$$

The codeword $\mathbf{c}'$ can be decoded via a simple two-step procedure. The decoder first reduces $\mathbf{y}''$ modulo-2. This eliminates the effect of the uncoded layers, and results in

$$\tilde{\mathbf{y}}^1 = \mathbf{y}'' \bmod 2 = [\mathbf{c}_b' + \mathbf{z}] \bmod 2.$$

Now, the coded layer $\mathbf{c}_b'$ can be decoded from $\tilde{\mathbf{y}}^1$. In the second step, the decoded layer $\mathbf{c}_b'$ is subtracted from $\mathbf{y}''$ giving rise to (assuming correct decoding of $\mathbf{c}_b'$)

$$\begin{aligned} \tilde{\mathbf{y}}^2 &= [\mathbf{y}'' - \mathbf{c}_b'] \bmod 2^L \\ &= \left[\sum_{l=1}^{L-1} 2^l \mathbf{c}_{u,l}' + \mathbf{z}\right] \bmod 2^L. \end{aligned}$$

The uncoded bits can be detected from $\tilde{\mathbf{y}}^2$ using a slicer with double step size as in Ungerboeck's set partitioning. For more details, see [34] and [35].

Due to the presence of the uncoded bits, the proposed code construction admits some error floor. If very low error probabilities are desired, one can extend the proposed construction to support multiple coded layers. This may be done by applying Construction D for lattices which is based on nested binary codes, where the nested binary codes should be cyclic. This construction, except for the cyclic codes constraint, is described in detail in [36] and [37].

VIII. Discussion and Conclusions

We have presented a novel DFE scheme for the discrete-time linear Gaussian channel, suitable for single-carrier transmission where CSI is not available at the transmitter. The scheme enables block decoding to be performed before applying the DFE when a cyclic code is used. The channel is equalized to an impulse response that is composed of integer coefficients only. The performance of the proposed scheme was analyzed, and in particular upper bounds on the induced noise enhancement were derived. Several examples of ISI channels were examined and it was shown that in many scenarios the scheme achieves a rather small gap-to-capacity. An interesting avenue for further research is to incorporate shaping into the scheme in order to make it attractive at low SNR. Another interesting question to be explored is how IF equalization can be combined with iterative equalization methods such as turbo equalization. Since the noise enhancement the FFE in IF equalization causes is always smaller than (or equal to) to that of a ZF-LE, combining it with an iterative equalizer may result in better convergence.

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