New Bounds on the Density of Lattice Coverings

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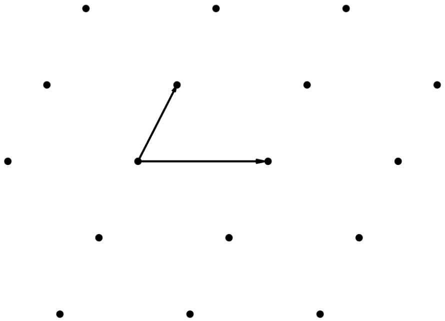
2020 Workshop on Coding, Cooperation, and Security in Modern Communication Networks (COCO 2020),

July 16, 2020

Definitions

- A **lattice** $L \subset \mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$
- It can be (non-uniquely) identified with a **generating matrix** $g = [g_1 | \cdots | g_n] \in \mathbb{R}^{n \times n}$, as

$$L = g\,\mathbb{Z}^n = \mathrm{span}_{\mathbb{Z}}(g_1, \ldots, g_n)$$



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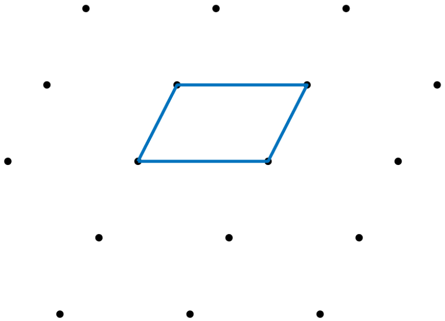

- **co-volume** of $L = |\det(G)|$ (volume of fundamental cell)

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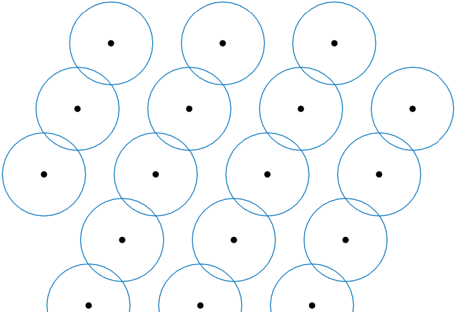

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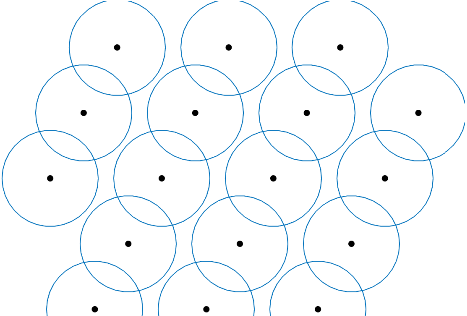

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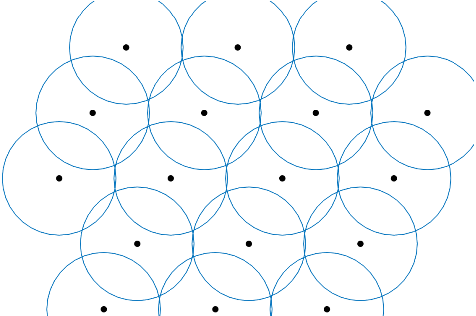

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- $\alpha L + \alpha r\mathcal{B} = \mathbb{R}^n \Leftrightarrow L + r\mathcal{B} = \mathbb{R}^n, \quad \forall \alpha \neq 0$
  $\Rightarrow$ Can assume WLOG that $L \in \mathcal{L}_n \triangleq \{g\,\mathbb{Z}^n \ : \ \det(g) = 1\}$

- The **optimal covering density** is

$$\Theta_n \triangleq \inf_{L \in \mathcal{L}_n} \Theta(L)$$

Definitions

More generally

- Let Conv_n denote the set of compact convex subsets of $\mathbb{R}^n$
- For $\mathcal{K} \in \mathrm{Conv}_n$ we define

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Note that clearly $\Theta_{n,\mathcal{K}} \geq 1$ for any $\mathcal{K}$

Previous Bounds

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- **Rogers'59,Coxeter-Few-Rogers'59:** $c_2 n < \Theta_n < n(\log(n))^{c_3}$
- **Gritzmann'85:** If $\mathcal{K} \in \mathrm{Conv}_n$ is "symmetric enough" $\Theta_{n,\mathcal{K}} < n(\log(n))^{c_3}$
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Our Theorem 1

$$\sup_{\mathcal{K} \in \mathrm{Conv}_n} \Theta_{n,\mathcal{K}} < cn^2$$

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- $\mathrm{SL}_n(\mathbb{R}) \triangleq \{g \in \mathbb{R}^{n\times n} \ : \det(g) = 1\}$
  $\mathrm{SL}_n(\mathbb{Z}) \triangleq \{t \in \mathbb{Z}^{n\times n} \ : \det(t) = 1\}$
- Any $L \in \mathcal{L}_n$ can be written as $L = g\,\mathbb{Z}^n$ for some $g \in \mathrm{SL}_n(\mathbb{R})$
- But, for any $t \in \mathrm{SL}_n(\mathbb{Z})$ we also have $L = g\ t\ \mathbb{Z}^n$
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A measure μ_n on $\mathcal{L}_n$ is $\mathrm{SL}_n(\mathbb{R})$-invariant if:

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There is a unique $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure μ_n on $\mathcal{L}_n$. We call it the Haar-Siegel measure.

Covering Density of a Typical Lattice

For $L \sim \mu_n$ the covering density $\Theta_{\mathcal{K}}(L)$ is a random variable
What can we say about it?

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- **Corollary from Rogers'58:** for any $\mathcal{K} \in \mathrm{Conv}_n$ and $1 < V < c_1 n$ we have $\Pr(\Theta_{\mathcal{K}}(L) > 2^n V) < c_2 e^{-V}$
- **Strömbergsson'12:** $\Pr(\Theta_{\mathcal{K}}(L) > (1.756\ldots)^n) < c_3 e^{-c_4 n}$

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For any $n \in \mathbb{N}$, $\mathcal{K} \in \mathrm{Conv}_n$ and $M \in [n^2, n^3]$ we have

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Covering Density of a Typical Lattice - Cont.

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Corollaries:

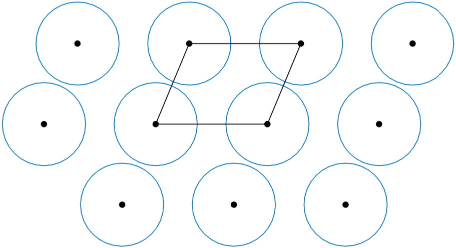
1. $\exists c > 0$ such that $\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Pr\left(\Theta_{\mathcal{K}}(L) > cn^2\right) < 1$
$\Longrightarrow \sup_{\mathcal{K}\in\mathrm{Conv}_n} \Theta_{n,\mathcal{K}} < cn^2$ which is our Theorem 1
2. $\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Pr\left(\Theta_{\mathcal{K}}(L) > \omega(n^2)\right) = o(1)$
3. $\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Pr\left(\Theta_{\mathcal{K}}(L) > cn^3\right) < c_7 e^{-c_8 n}$
Improves previous result of Strömbergsson with n^3 instead of $(1.756\ldots)^n$.

Thm 1 : Proof Outline

We dilate $\mathcal{K}$ to have volume $1 < V < cn$ (WLOG)

Step 1 - almost perfect covering: Find a lattice $L \in \mathcal{L}_n$ such

that at least $1 - \epsilon$ of the fundamental cell (torus) is covered
Rogers'58: if $L \sim \mu_n$, this happens with probability $\gtrapprox 1 - \epsilon^{-1}e^{-V}$



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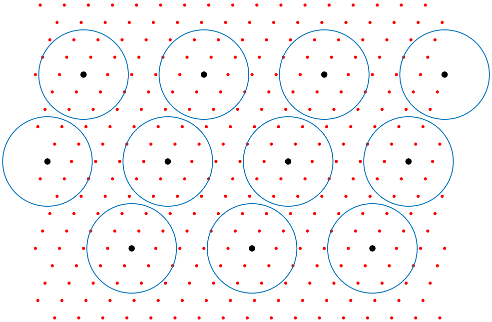

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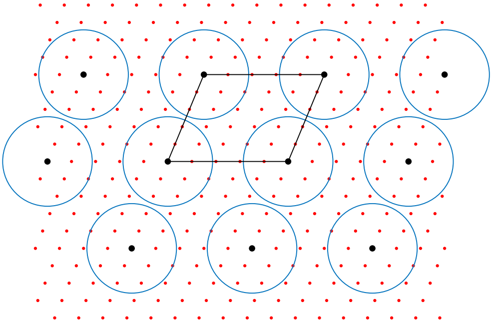

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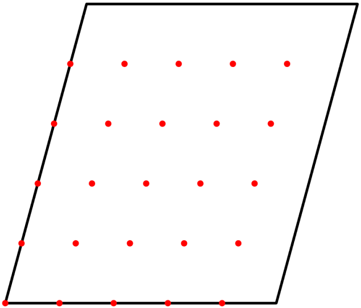

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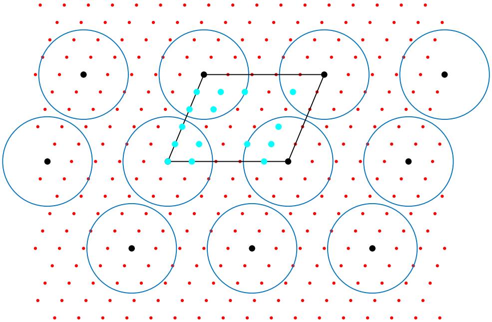

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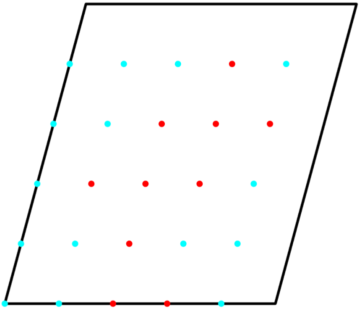

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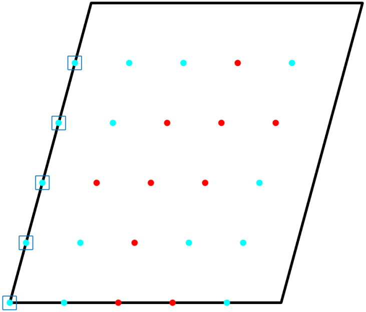

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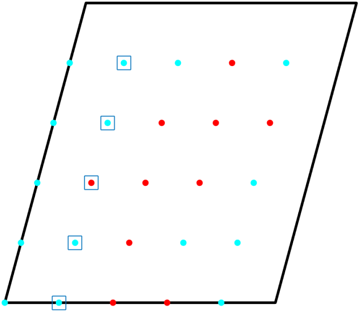

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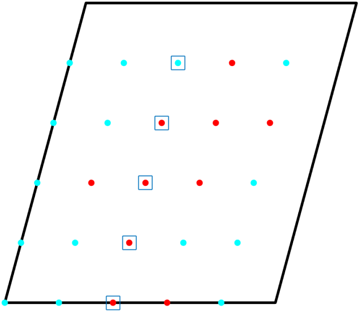

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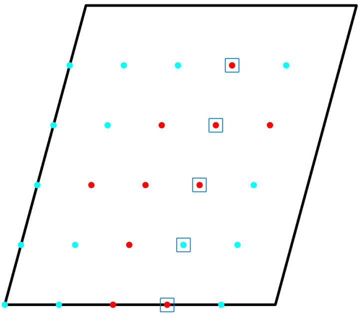

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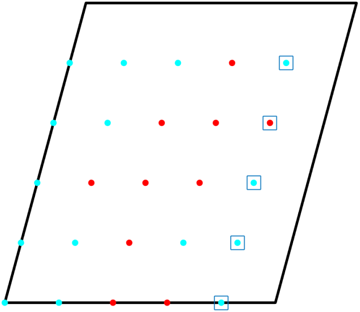

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Theorem [Dvir,Kopparty,Lev,Saraf,Sudan 2009-2013]

If $|\mathcal{A}^c| < e^{-n/p} p^n$ it is not a Kakeya-set of rank 2.

For our probabilistic theorem, we prove a soft version of this
Recall that $|\mathcal{A}^c| < \epsilon p^n$, thus if $\epsilon < e^{-1}$ we can find such S

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Step 4 - Use S to kill the discrete hole:

Construct a lattice L_1, such that $L \subset L_1 \subset L_2$ and $L_1/L_2 \cong S$

In particular, if g is the natural mapping from $\mathbb{F}_p^n$ to the fundamental cell

$$L_1 = g(S) + L$$

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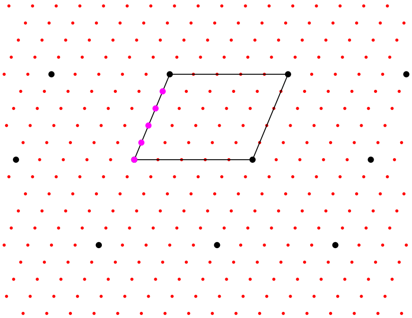

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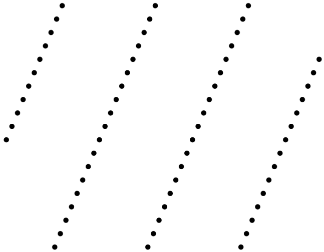

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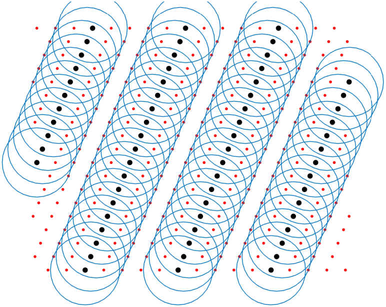

The good: $L_2 \subset L_1 + \mathcal{K}$, The bad: $\frac{\text{co-volume of}(L_1)}{\text{co-volume of}(L)} = p^{-r}$

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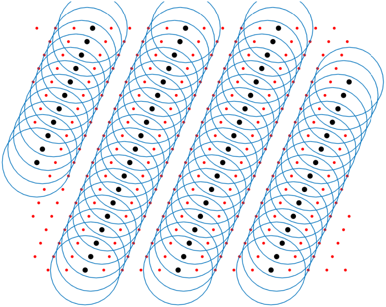

Remark: If $L \sim \mu_n$ and S is a a uniform subspace, then $p^{r/n} L_1 \sim \mu_n$

Thm 1 : Proof Outline

Step 5 - From covering L_2 to covering $\mathbb{R}^n$

Fact:
If $L + \mathcal{K}$ covers more than 1/2 of the space, then $L + 2\mathcal{K} = \mathbb{R}^n$

$\implies L_2 + \frac{2}{p}\mathcal{K} = \mathbb{R}^n$ (since $L_2 = \frac{1}{p}L$)

$\implies L_1 + \mathcal{K} + \frac{2}{p}\mathcal{K} = \mathbb{R}^n$ (since $L_2 \subset L_1 + \mathcal{K}$)

$\implies L_1 + \left(1 + \frac{2}{p}\right)\mathcal{K} = \mathbb{R}^n$ (since $\mathcal{K}$ is convex)

$\implies p^{r/n}L_1 + p^{r/n}\left(1 + \frac{2}{p}\right)\mathcal{K} = \mathbb{R}^n$, and $p^{r/n}L_1 \in \mathcal{L}_n$

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We used $r = 2$, $n < p < 2n$ and all arguments hold with $V = \text{const}$
$\implies \Theta_{\mathcal{K}}(p^{r/n}L_1) < cn^2$

Thanks for your attention!