Bounds on the density of nearly uniform lattice coverings

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Joint work with Oded Regev (NYU) and Barak Weiss (TAU)

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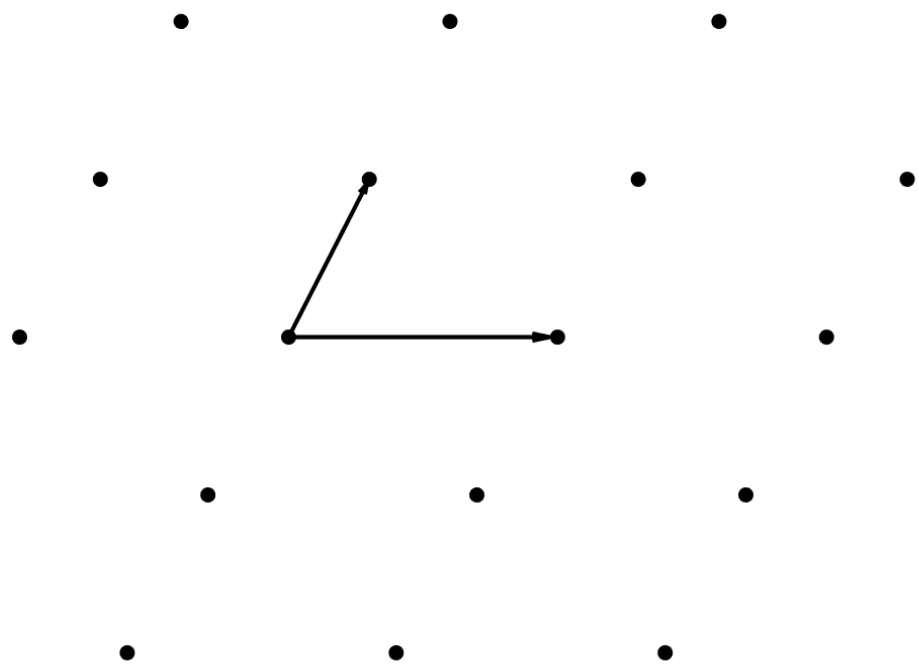
Definitions

- A **lattice** $L \subset \mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$
- It can be (non-uniquely) identified with a full-rank **generating matrix** $g = [g_1 | \cdots | g_n] \in \mathbb{R}^{n \times n}$, as

$$L = g\,\mathbb{Z}^n = \mathrm{span}_{\mathbb{Z}}(g_1, \ldots, g_n)$$



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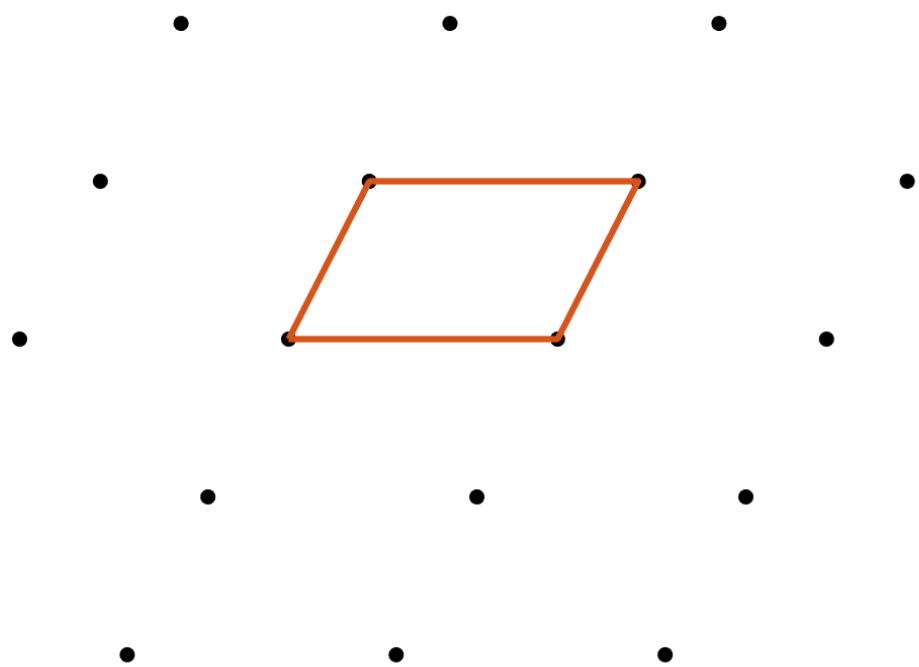

- **co-volume** of $L = |\det(g)|$ (volume of fundamental cell)

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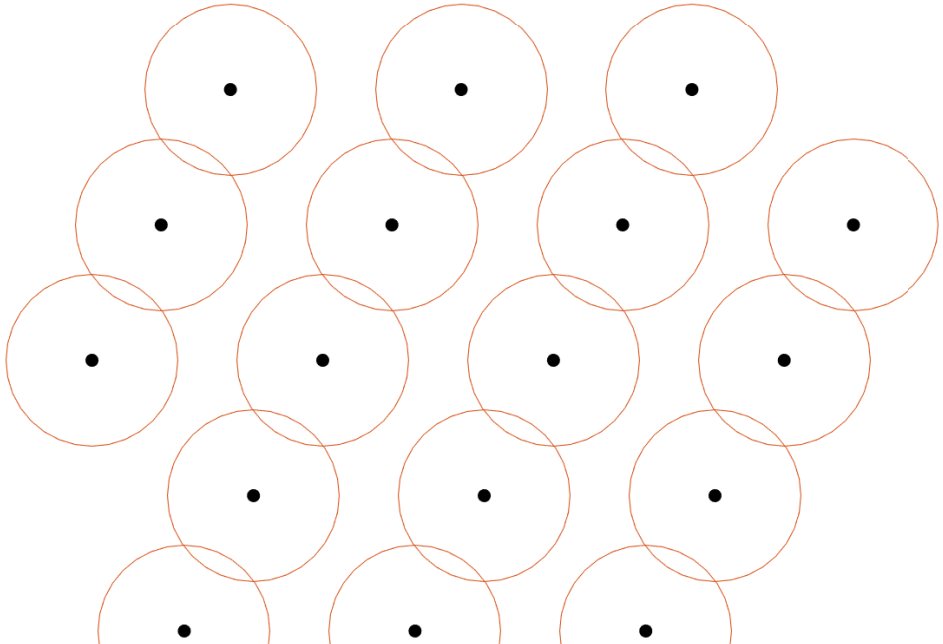

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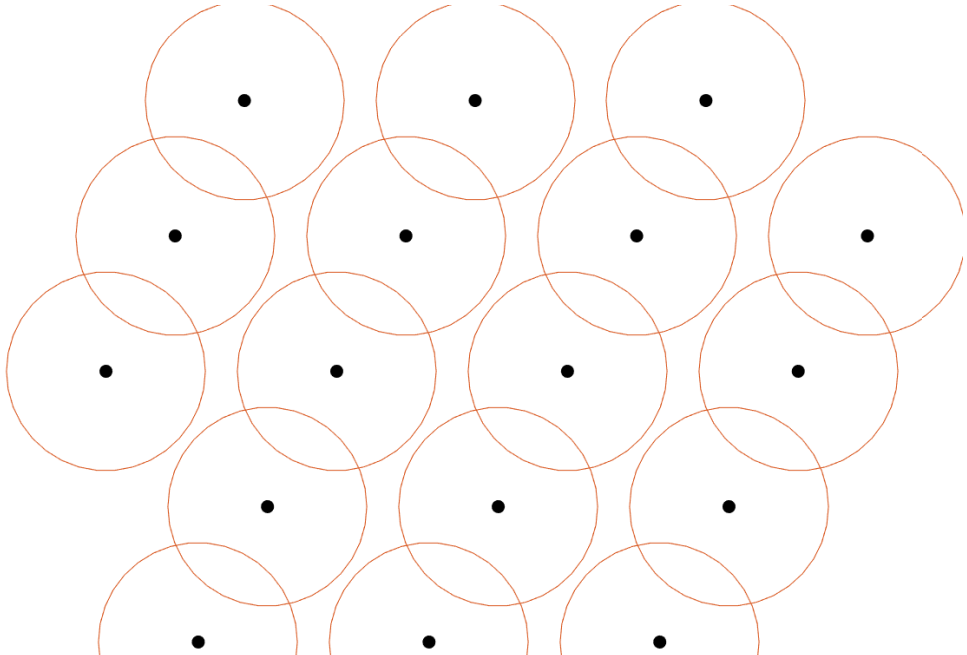

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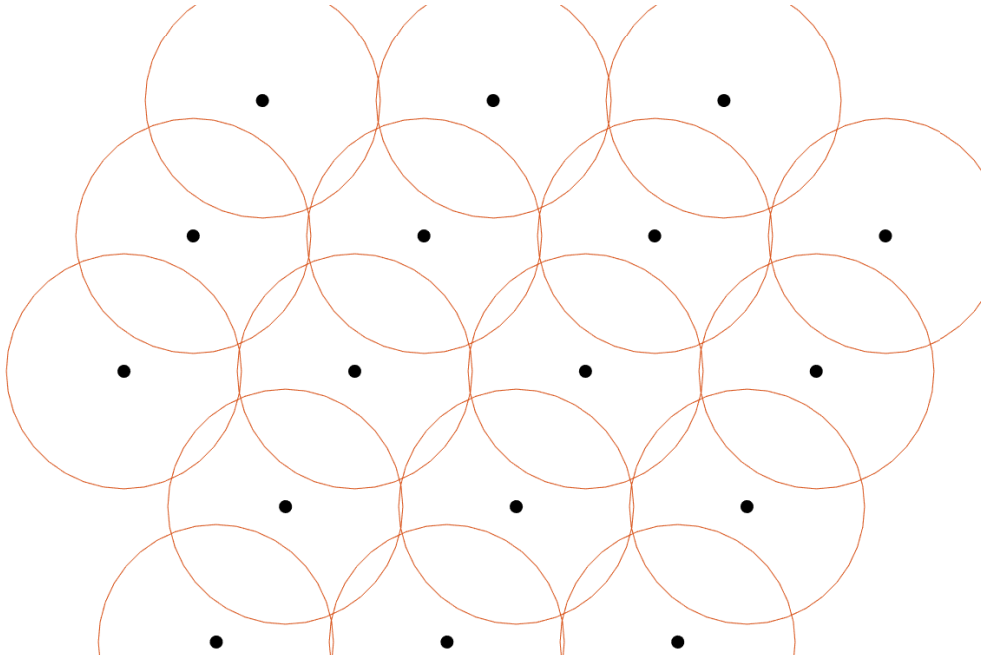

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- $\alpha L + \alpha r\mathcal{B} = \mathbb{R}^n \Leftrightarrow L + r\mathcal{B} = \mathbb{R}^n, \quad \forall \alpha \neq 0$
  $\Rightarrow$ Can assume WLOG that $L \in \mathcal{L}_n \triangleq \{g\,\mathbb{Z}^n \ : \ \det(g) = 1\}$

- The **optimal covering density** is

$$\Theta_n \triangleq \inf_{L \in \mathcal{L}_n} \Theta(L)$$

Definitions

More generally

- Let Conv_n denote the set of compact convex subsets of $\mathbb{R}^n$
- For $\mathcal{K} \in \mathrm{Conv}_n$ we define

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- Many applications in EECS: placing cellular basestations ($n = 3$), lossy compression ($n \gg 1$), enforcing power constraint in communication, derivation of upper/lower bounds through covering numbers...

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- **Rogers'59,Coxeter-Few-Rogers'59:** $c_2 n < \Theta_n < n(\log(n))^{c_3}$
- **Gritzmann'85:** If $\mathcal{K} \in \mathrm{Conv}_n$ is "symmetric enough" $\Theta_{n,\mathcal{K}} < n(\log(n))^{c_3}$
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Theorem 1 [O.-Regev-Weiss, JAMS22]

$$\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Theta_{n,\mathcal{K}} < cn^2$$

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Recall:

- $\mathrm{SL}_n(\mathbb{R}) \triangleq \{g \in \mathbb{R}^{n\times n} : \det(g) = 1\}$
  $\mathrm{SL}_n(\mathbb{Z}) \triangleq \{t \in \mathbb{Z}^{n\times n} : \det(t) = 1\}$
- Any $L \in \mathcal{L}_n$ can be written as $L = g\,\mathbb{Z}^n$ for some $g \in \mathrm{SL}_n(\mathbb{R})$
- But, for any $t \in \mathrm{SL}_n(\mathbb{Z})$ we also have $L = g\,t\,\mathbb{Z}^n$
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A measure μ_n on $\mathcal{L}_n$ is $\mathrm{SL}_n(\mathbb{R})$-invariant if:

$$\mu_n(gE) = \mu_n(E) \ : \forall E \subset \mathcal{L}_n, \forall g \in \mathrm{SL}_n(\mathbb{R})$$

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There is a unique $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure μ_n on $\mathcal{L}_n$. We call it the Haar-Siegel measure.

Covering Density of a Typical Lattice

For $L \sim \mu_n$ the covering density $\Theta_{\mathcal{K}}(L)$ is a random variable
What can we say about it?

- Easy fact (from [O.-Regev-Weiss, JAMS'22]): $\mathbb{E}[\Theta_{\mathcal{K}}(L)] = \infty$

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What was known? **Strömbergsson'12:**

- Easy corollary from Rogers'58: for any $\mathcal{K} \in \mathrm{Conv}_n$ and $1 < V < c_1 n$ we have $\Pr(\Theta_{\mathcal{K}}(L) > 2^n V) < c_2 e^{-V}$
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Theorem 2 [O.-Regev-Weiss, JAMS'22]

For any $n \in \mathbb{N}$, $\mathcal{K} \in \mathrm{Conv}_n$ and $M \in [n^2, n^3]$ we have

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Covering Density of a Typical Lattice - Cont.

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Corollaries:

1. $\exists c > 0$ such that $\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Pr\left(\Theta_{\mathcal{K}}(L) > cn^2\right) < 1$
   $\Longrightarrow \sup_{\mathcal{K}\in\mathrm{Conv}_n} \Theta_{n,\mathcal{K}} < cn^2$ which is our Theorem 1
2. $\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Pr\left(\Theta_{\mathcal{K}}(L) > \omega(n^2)\right) = o(1)$
3. $\sup_{\mathcal{K}\in\mathrm{Conv}_n} \Pr\left(\Theta_{\mathcal{K}}(L) > cn^3\right) < c_7 e^{-c_8 n}$
   Improves previous result of Strömbergsson with n^3 instead of $(1.756\ldots)^n$.

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- The number of times a point is covered is denoted by

$$N(L, \mathcal{K}, x) \stackrel{\text{def}}{=} |(L - x) \cap \mathcal{K}| = |\{y \in L \ : \ x \in y - \mathcal{K}\}|$$

- "On average" a point is covered $\mathrm{vol}(r\mathcal{K})/\mathrm{covol}(L)$ times:

$$\mathbb{E}_{X \sim \mathrm{Unif}(\mathbb{R}^n/L)} N(L, r\mathcal{K}, X) = \mathrm{vol}(r\mathcal{K})/\mathrm{covol}(L)$$

- Furthermore, for all $x \in \mathbb{R}^n$ we have that

$$\lim_{r \to \infty} \frac{N(L, r\mathcal{K}, x)}{\mathrm{vol}(r\mathcal{K})/\mathrm{covol}(L)} = 1$$

Nearly Uniform Covering - Cont.

How large does $\mathrm{vol}(r\mathcal{K})$ have to be for $\frac{N(L,r\mathcal{K},x)}{\mathrm{vol}(r\mathcal{K})/\mathrm{covol}(L)} \approx 1$ to hold for all $x \in \mathbb{R}^n$?

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Definitions:

- Covering smoothness:

$$\eta(r\mathcal{K}, L) \stackrel{\mathrm{def}}{=} \sup_{x\in\mathbb{R}^n} \left| \frac{N(L, r\mathcal{K}, x)}{\mathrm{vol}(r\mathcal{K})/\mathrm{covol}(L)} - 1 \right|$$

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- Since $r \mapsto \eta(r\mathcal{K}, L)$ is not monotone, we also define

$$\Phi_{\mathcal{K},L}(\varepsilon) \stackrel{\mathrm{def}}{=} \sup \left\{ \frac{\mathrm{vol}(r\mathcal{K})}{\mathrm{covol}(L)} \ : \ r > 0 \text{ satisfies } \eta(r\mathcal{K}, L) > \varepsilon \right\}.$$

Nearly Uniform Covering: Results for μ_n

Theorem [O.-Regev-Weiss'23]

Let $n > 25$, let $\mathcal{K} \in \mathrm{Conv}_n$, and let $\delta, \varepsilon \in (0,1)$ and assume $\mathrm{vol}(\mathcal{K}) > c_1 \left(\frac{1}{\varepsilon\delta}\right)^{6.5} n^3$. Then, for $L \sim \mu_n$ we have

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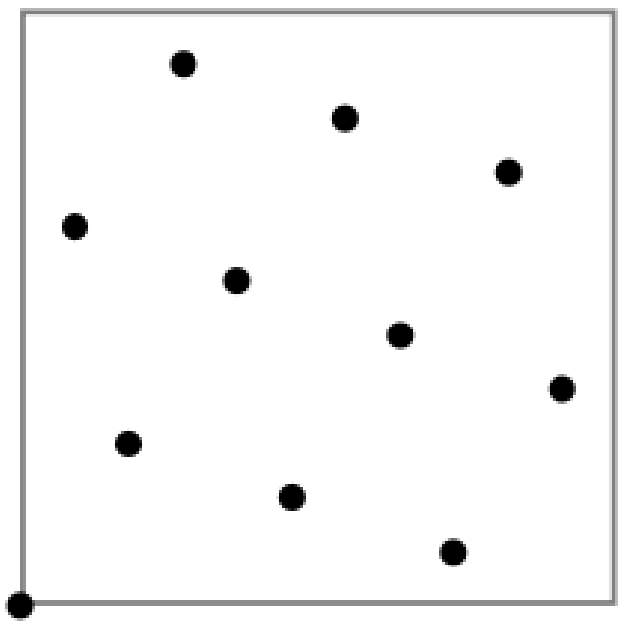
Nearly Uniform Covering: Results for Construction A

- For a prime p denote by $\mathrm{Gr}_{n,r}(\mathbb{F}_p)$ the collection of subspaces of dimension r in $\mathbb{F}_p^n$
- $\pi_p : \mathbb{Z}^n \to \mathbb{F}_p^n$ is the natural mapping, and π_p^{-1} its inverse
- For $S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)$ the corresponding construction A lattice is $L = \pi_p^{-1}(S)$



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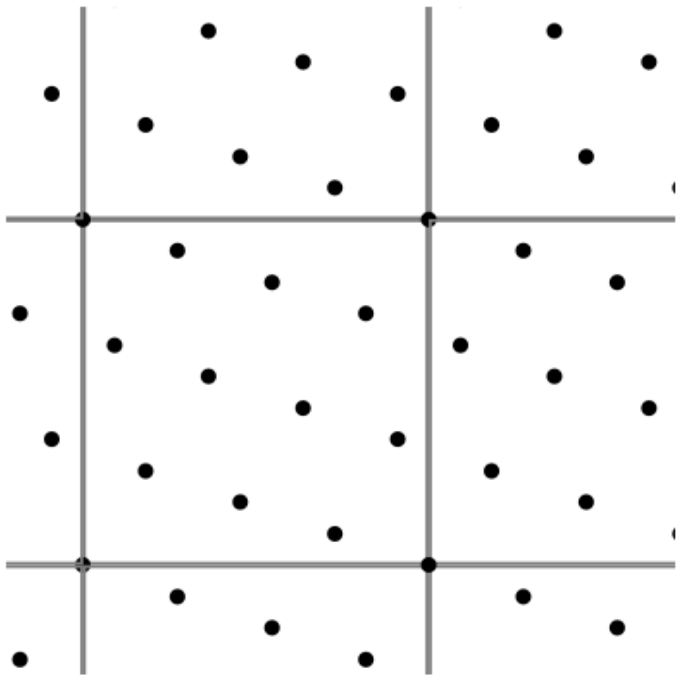

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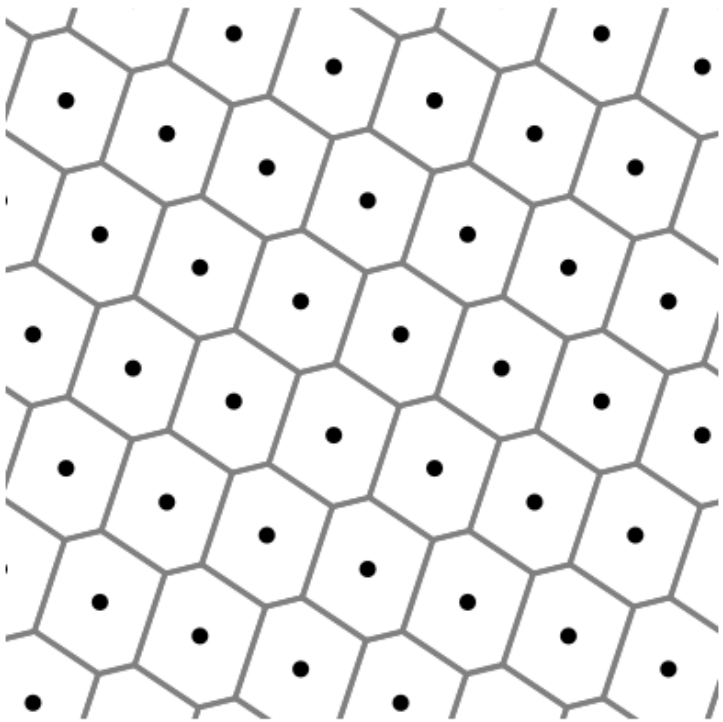

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$$\frac{1024}{(\varepsilon\delta)^2} n^2 \leq p \leq \frac{2048}{(\varepsilon\delta)^2} n^2,$$

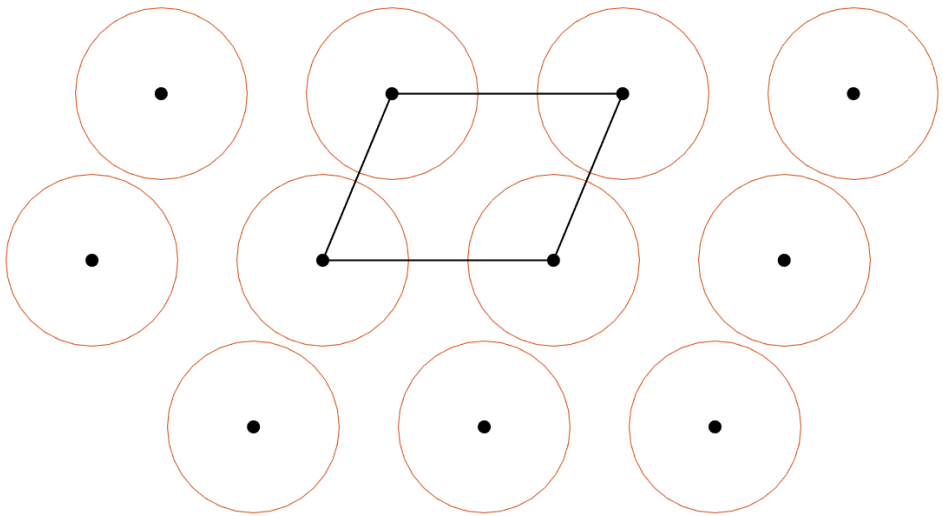
and $r = 3 + \left\lceil \frac{n}{\log p} \left(\frac{1}{2} \log 9n\right) \right\rceil$. Then if L is drawn from the (p, r) random construction A ensemble (so that $\mathrm{covol}(p^{r/n} L) = 1$), we have

$$\Pr\left(\eta(\mathcal{B}, p^{r/n} L) > \varepsilon\right) \leq \delta.$$

Main Thm : Proof Outline

Step 1 - packing and "rough" covering:

- We dilate $\mathcal{K}$ such that $\mathrm{vol}(\mathcal{K}) = 5^{-n}$
  Thus, $\mathrm{vol}(\mathcal{K} - \mathcal{K}) < (4/5)^n < 1$ (Rogers-Shephard'57)
- By Siegel's summation formula, if $L \sim \mu_n$ it forms a packing w.r.t. $\mathcal{K}$ w.h.p.
- Furthermore, $L + 11\mathcal{K} = \mathbb{R}^n$ w.h.p. $(\mathrm{vol}(11\mathcal{K}) > 2^n)$



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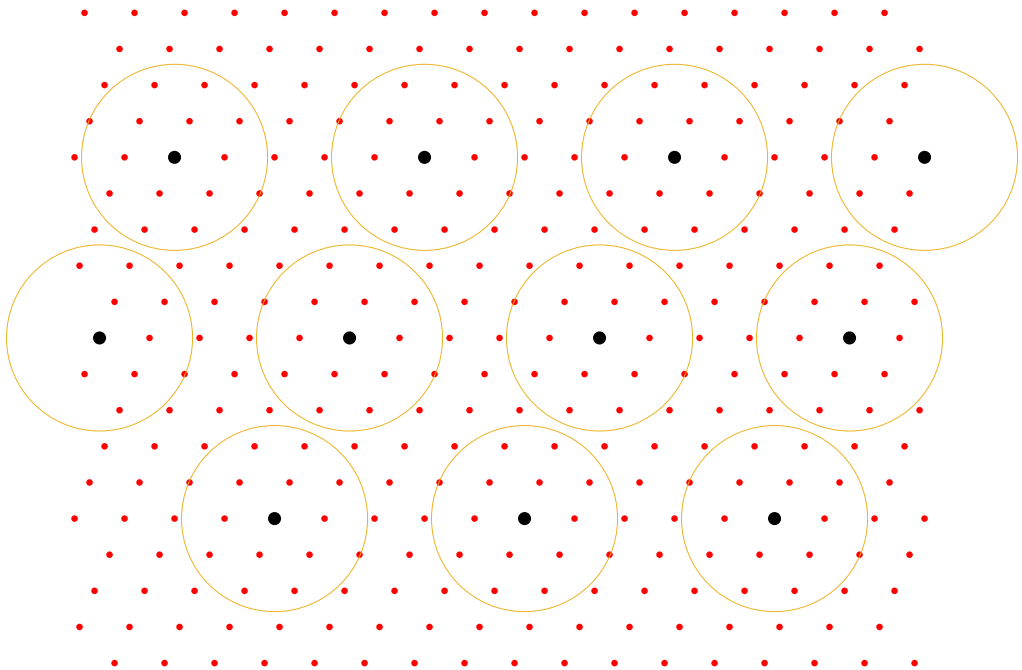

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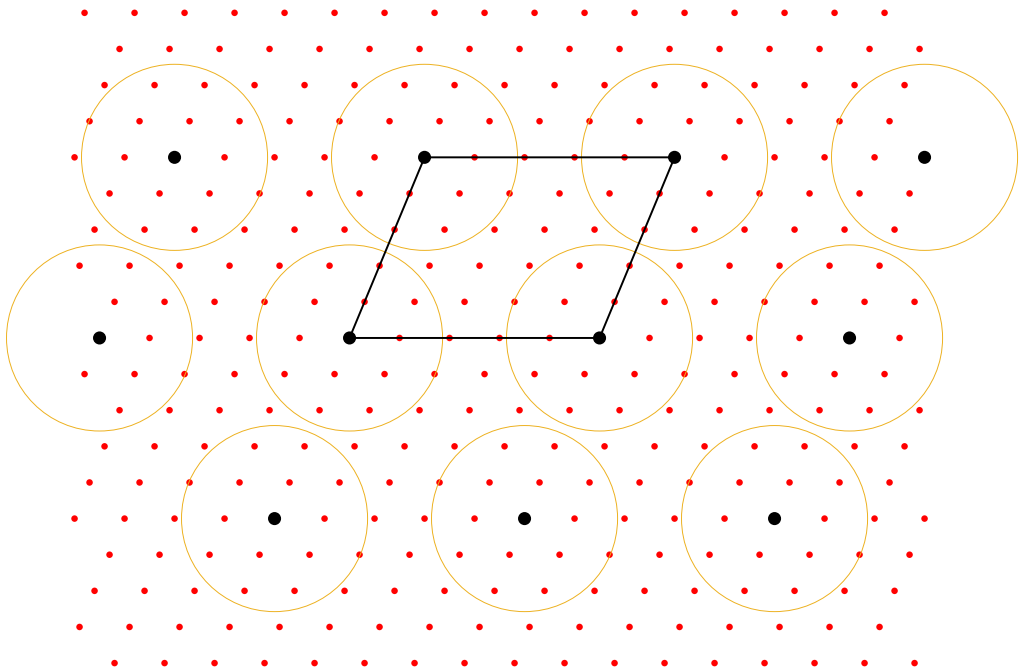

$L_2/L \cong \mathbb{F}_p^n$

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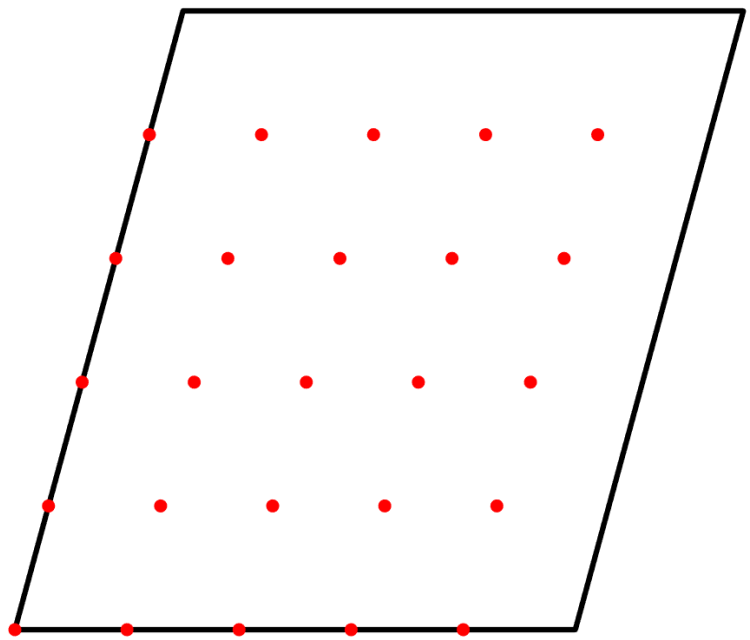

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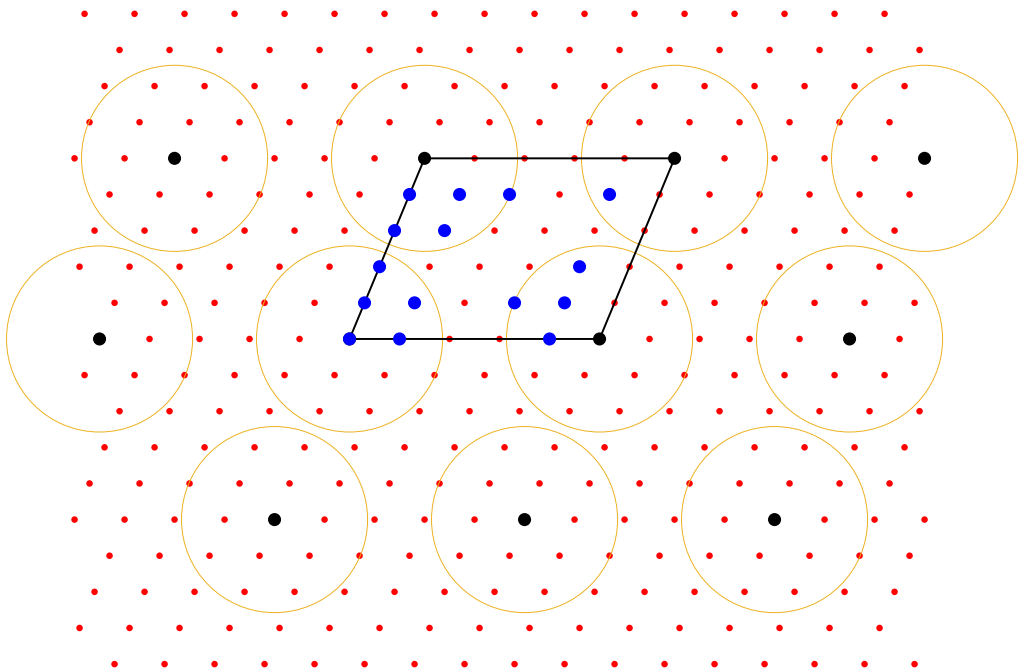

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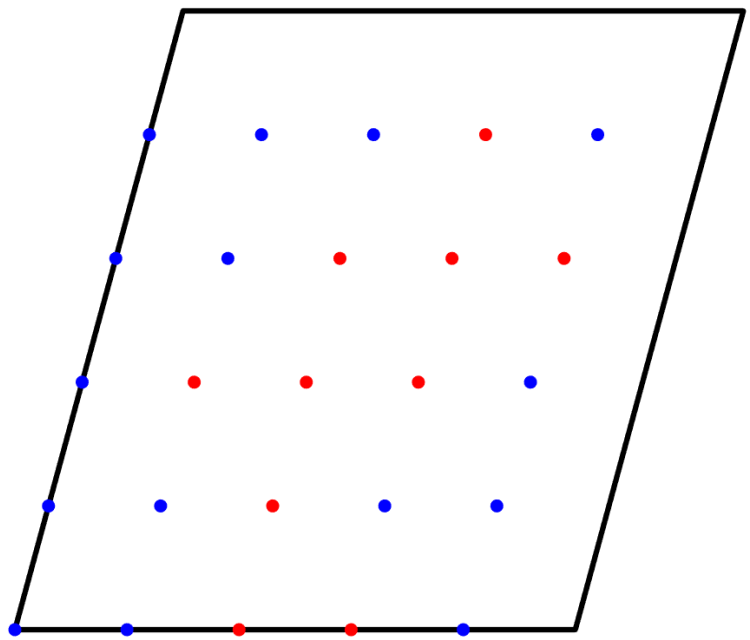

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Choose some subspace $S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)$, such that

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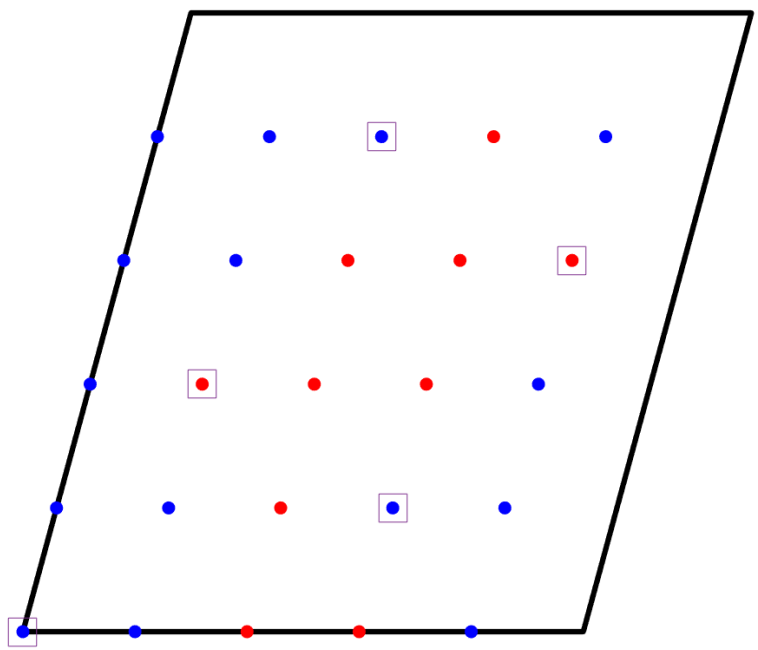

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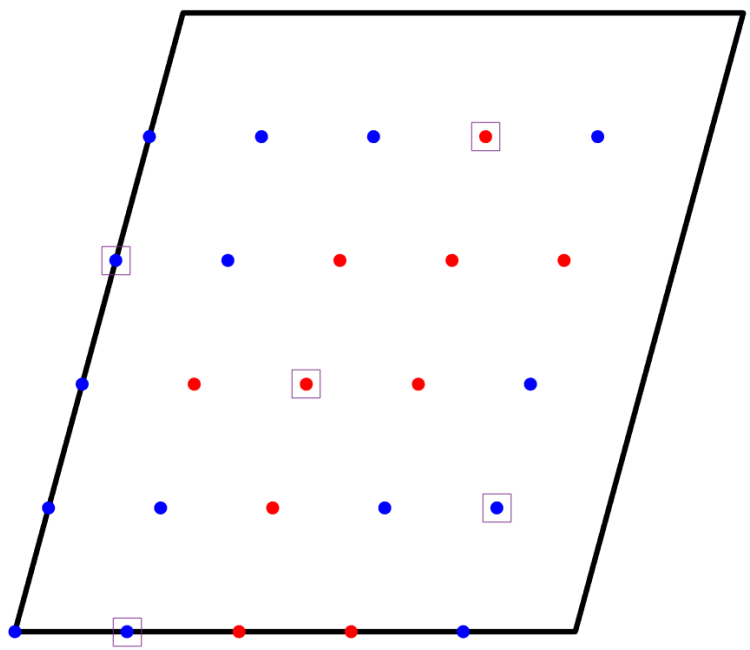

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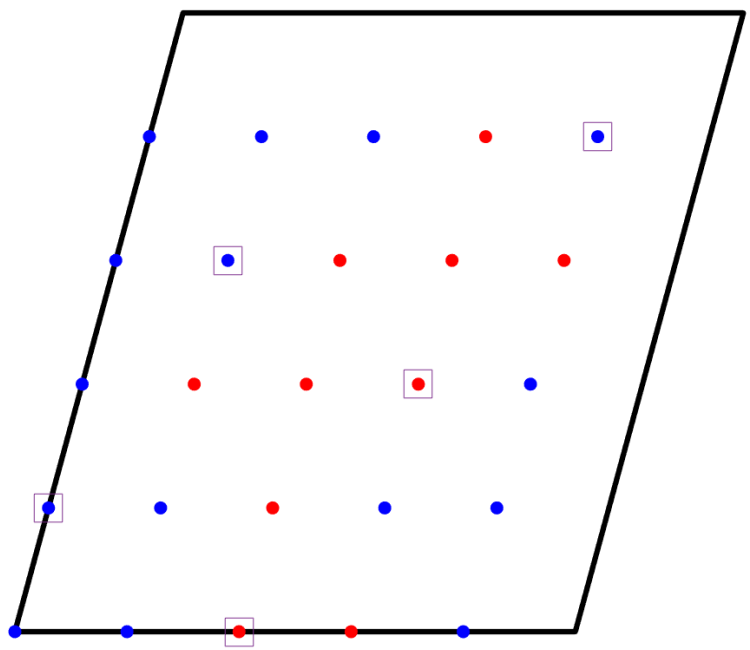

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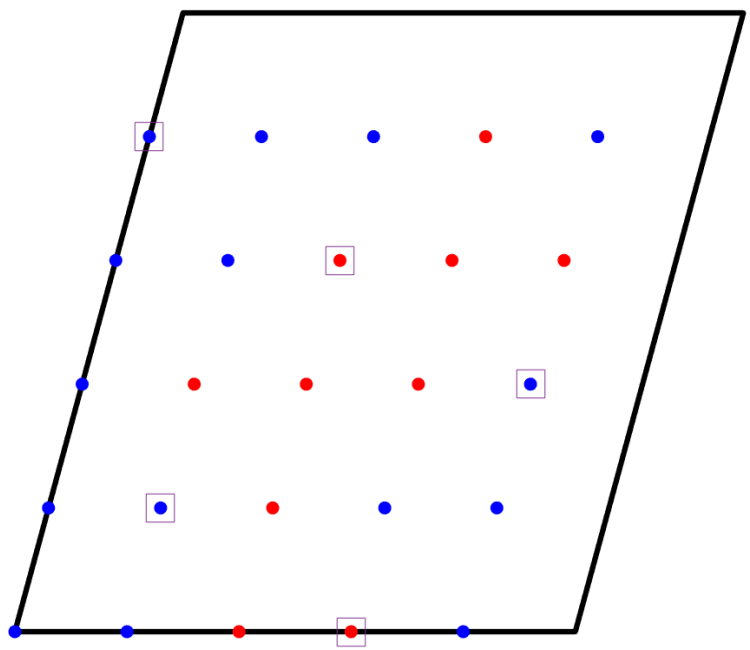

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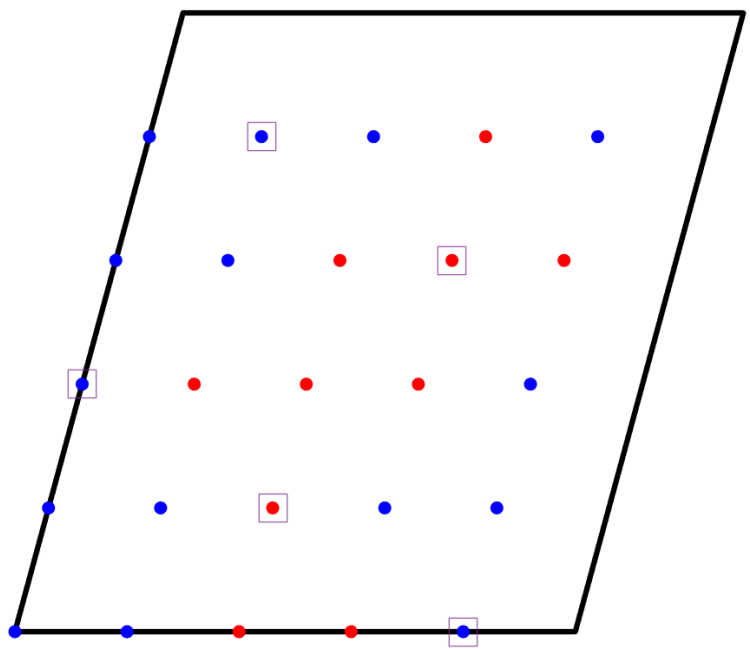

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How large should r be for this to be possible?
Clearly $r = n$ works, and we must have $|\mathcal{A}| p^r > p^n$

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Theorem [Dhar-Dvir'22]

For any $\mathcal{A} \subset \mathbb{F}_p^n$ and integer $4 \leq r \leq n$ chosen such that $|\mathcal{A}| \cdot p^r > p^{n+3}$, a random subset $S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)$ smooths $\mathcal{A}$ w.h.p. provided that $p = \omega(n)$

Main Thm : Proof Outline

Step 3 - discrete smoothing:

Choose some subspace $S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)$, such that

$$\forall x \in \mathbb{F}_p^n \ : |(x+S) \cap \mathcal{A}| \in (1 \pm \varepsilon)|\mathcal{A}|p^{r-n}$$

More formally:

Theorem [Dhar-Dvir'22]

Let

- $n \geq 5$, $\delta, \varepsilon \in (0,1)$, and $p > 64n/(\varepsilon\delta)^2$
- $A \subset \mathbb{F}_p^n$, $|A| > p^4$ and $r > 3 + n - \log_p |A|$
- $S \sim \mathrm{Unif}\left(\mathrm{Gr}_{n,r}(\mathbb{F}_p)\right)$

Then,

$$\Pr\left(\max_{x \in \mathbb{F}_p^n} \left| \frac{|(x+S) \cap \mathcal{A}|}{|\mathcal{A}|p^{r-n}} - 1 \right| > \varepsilon \right) < \delta.$$

Main Thm : Proof Outline

Step 4 - Create a denser lattice

Construct a lattice $L(S)$, such that $L \subset L(S) \subset \frac{L}{p}$ and $L(S)/\frac{L}{p} \cong S$

In particular, if g is the natural mapping from $\mathbb{F}_p^n$ to the fundamental cell

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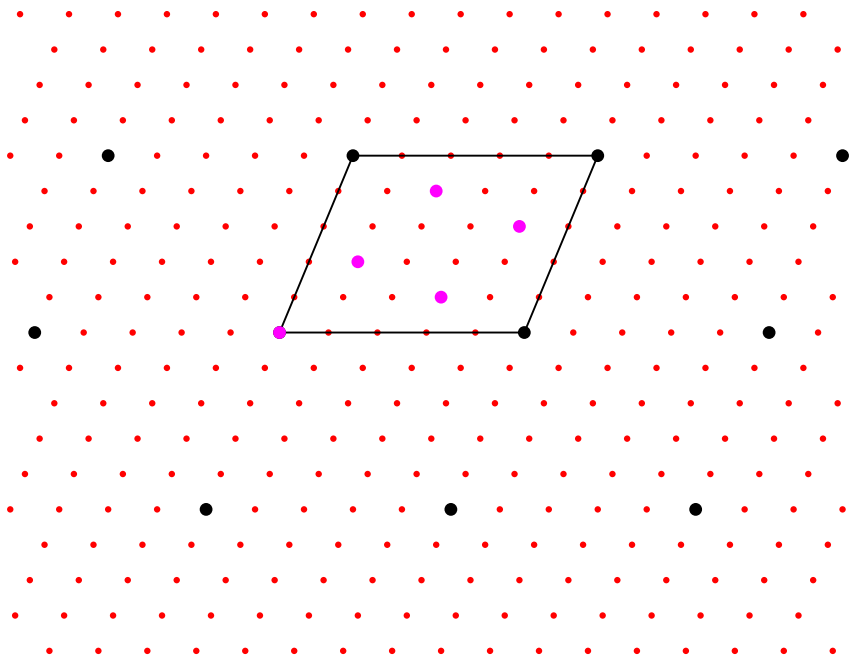

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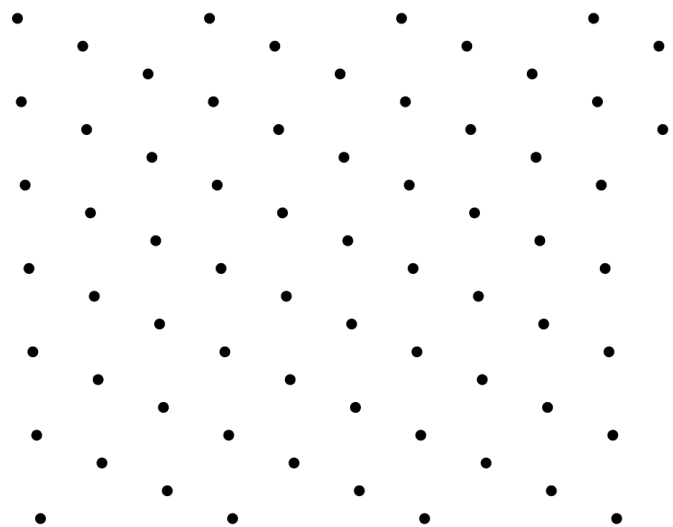

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Proposition [O.-Regev-Weiss, JAMS'22]

If $L \sim \mu_n$ and $S \sim \mathrm{Unif}(\mathrm{Gr}_{n,r}(\mathbb{F}_p))$ are statistically independent, then $\alpha L(S) \sim \mu_n$ for $\alpha = p^{r/n}$.

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Proof: for any $g' \in \mathrm{SL}_n(\mathbb{R})$ we have

$$g' \cdot \{L(S) \ : \ S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)\} = \{(g'L)(S) \ : \ S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)\}$$

Thus, the probability measure governing $L(S)$ is $\mathrm{SL}_n(\mathbb{R})$-invariant

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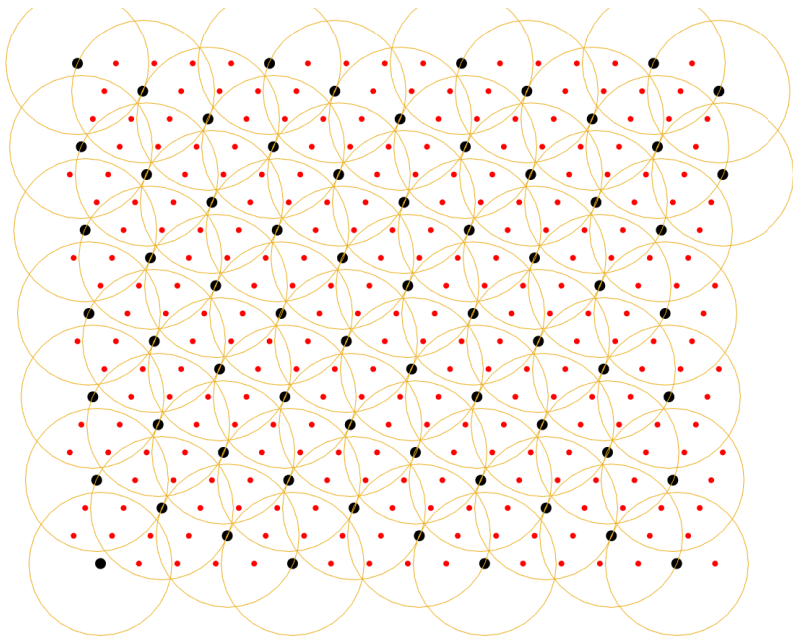

The points of $\frac{L}{p}$ are almost uniformly covered by $L(S) + \mathcal{K}$ w.h.p. over $L \sim \mu_n$ and $S \sim \mathrm{Unif}(\mathrm{Gr}_{n,r})$ [r chosen according to DD'22]

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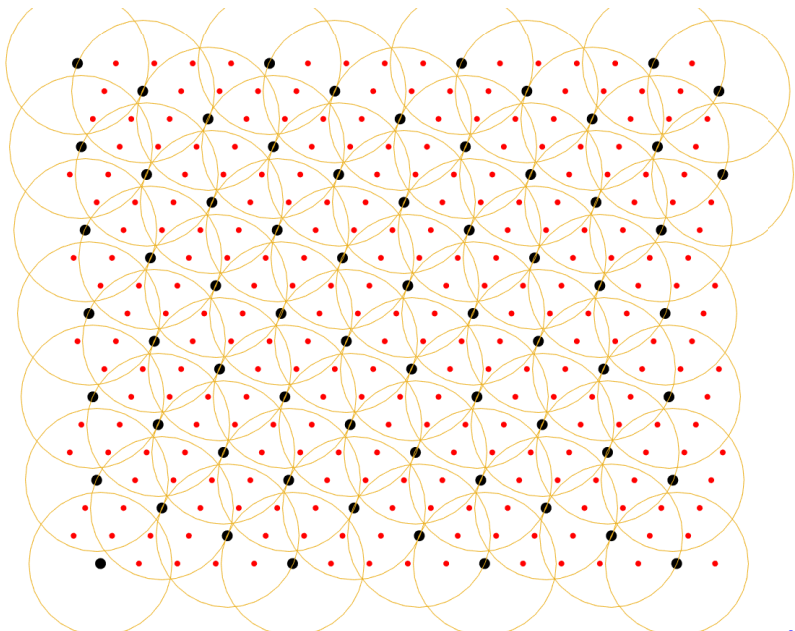

The points of $\frac{L}{p}$ are almost uniformly covered by $\tilde{L} + p^{r/n}\mathcal{K}$ w.h.p. over $\tilde{L} \sim \mu_n$ [r chosen according to DD'22]

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- Recall that by DD'22 $r \approx 3 + n - \log_p |\mathcal{A}|$ and $p = \Omega(n)$ suffice
- $|\mathcal{A}| \approx \mathrm{vol}(\mathcal{K})p^n$

Thus, we can choose p and r such that almost uniform covering is obtained w.h.p. and

$$\mathrm{vol}\left(p^{r/n}\mathcal{K}\right) = p^r \mathrm{vol}(\mathcal{K}) = O(n^3)$$

Main Thm : Proof Outline

Step 5 - From uniformly covering $\frac{L}{p}$ to uniformly covering $\mathbb{R}^n$

Lemma

Let $L, L' \subset \mathbb{R}^n$ be lattices, $\mathcal{K} \subset \mathbb{R}^n$ be a convex set, and assume $0 < \rho < 1$ is such that $L' + \rho\mathcal{K} = \mathbb{R}^n$. Then:

1. $\max_{x \in \mathbb{R}^n} N(L, \mathcal{K}, x) \leq \max_{x' \in L'} N(L, (1+\rho)\mathcal{K}, x')$;
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We apply this lemma with $L' = \frac{L}{p}$, and $\rho = \frac{11}{p} = O(1/n)$. Details are omitted.

Thanks for your attention!