Minimax Risk Upper Bounds Based on Shell Analysis of a Quantized Maximum Likelihood Estimator

Or Ordentlich
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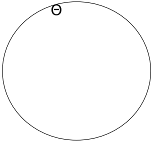
Algorithmic Structures for Uncoordinated Communications and Statistical Inference in Exceedingly Large Spaces
BIRS, Banff
March 15, 2024

Problem Setting: High Dimensional Parameter Estimation

- Parameter space: $\Theta \subset \mathbb{R}^d$, loss $\ell : \Theta \times \Theta \to \mathbb{R}_+$



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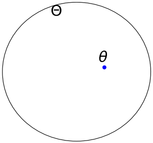

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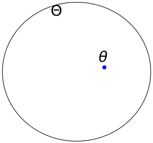

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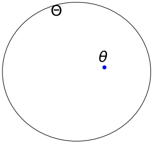
Objective

Upper bound the minimax risk

$$\min_{\hat{\theta}(\cdot)} \max_{\theta} \mathbb{E}_{Y^n \sim P_\theta} \left[\ell(\theta, \hat{\theta}(Y^n)) \right]$$

or its PAC proxy

$$\min_{\hat{\theta}(\cdot)} \max_{\theta} P_\theta \left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta \right]$$



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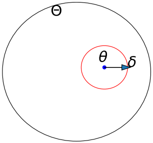

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Our objective

Develop a **unified** information-theoretic framework for **upper bounding**

$$\min_{\hat{\theta}(\cdot)} \max_{\theta} P_\theta \left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta \right]$$

Reminder: Unified IT framework for lower bounds

Deriving a general **lower bound** on minimax risk is easy

Mutual Information Metohd [Polyanskiy-Wu, Chapter 30]

Fix a prior $\theta \sim \pi$

$$\theta \overset{P_{Y^n|\theta}}{—} Y^n — \hat{\theta}$$

Examine:

$$I\left(\theta; \hat{\theta}(Y^n)\right)$$

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Let $R(D)$ be the RDF for π, ℓ. By definition

$$R\left(\mathbb{E}\left[\ell(\theta, \hat{\theta})\right]\right) \leq I\left(\theta; \hat{\theta}(Y^n)\right)$$

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Data Processing Inequality gives

$$R\left(\mathbb{E}\left[\ell(\theta, \hat{\theta})\right]\right) \leq I\left(\theta; \hat{\theta}(Y^n)\right) \leq I\left(\theta; Y^n\right)$$

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By definition of capacity

$$R\left(\mathbb{E}\left[\ell(\theta,\hat{\theta})\right]\right) \leq I\left(\theta;\hat{\theta}(Y^n)\right) \leq I\left(\theta;Y^n\right) \leq C(P_{Y^n|\theta})$$

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- Tight in many cases (for "good" choice of π)
- Lower bound decouples analysis of Θ and sample model

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- Decoupling has a price - the output of $P_{Y^n|\theta}$ can provide much information on θ that is not helpful for estimation under the $\ell(\cdot,\cdot)$ distortion measure
  $\Theta = \mathcal{S}^{d-1}$, $Y^n = \theta + \sigma Z$, $\ell(\theta,\hat{\theta}) = \sum_{i=1}^{d} |[\theta_i - \hat{\theta}_i] \bmod \epsilon|$

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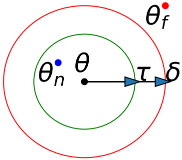
- We characterize the "sensitivity" of P_θ to large ℓ-variations in θ via mismatched binary hypothesis testing

Main Result - Definitions



- Let $\tau > 0$
- $\Theta \subset \mathbb{R}^d$, $\ell(\theta, \hat{\theta}) = \|\theta - \hat{\theta}\|$ for some norm, $\mathrm{diam}(\Theta) = e^{e^{o(d)}}$
- $\theta, \theta_n, \theta_f \in \Theta$
  - $Y^n \sim P_\theta$
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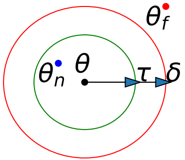

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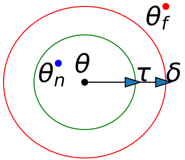

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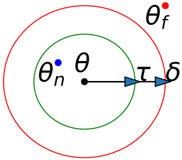

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- Worst case error exponent w.r.t $\theta, \theta_n, \theta_f$

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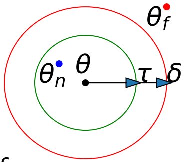

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Def: Hypothesis Testing Error Exponent (Mismatched,Worst Case)

$$E_{HT}(\tau, \delta) \triangleq \min_{\substack{\theta,\theta_n,\theta_f \in \Theta \\ \ell(\theta,\theta_n) \leq \tau \\ \ell(\theta,\theta_f) \geq \delta}} -\frac{1}{d} \log P_\theta \left[\frac{dP_{\theta_f}}{dP_{\theta_n}}(Y^n) \geq 1 \right]$$

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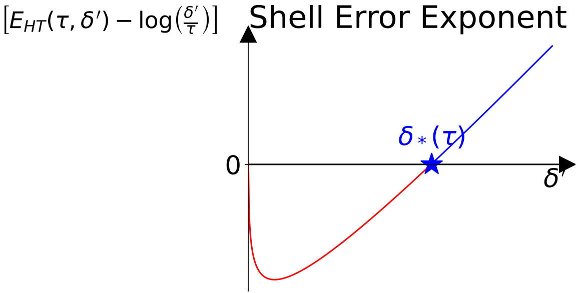

Def: Critical Loss

$$\delta_*(\tau) \triangleq \sup\left\{\delta' > 0 : \left[E_{HT}(\tau,\delta') - \log\frac{\delta'}{\tau}\right] \leq 0\right\}$$

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To use theorem: bound $E_{HT}(\tau,\delta)$, choose τ

$\hat{\theta} \triangleq$ Quantized Maximum Likelihood Estimator

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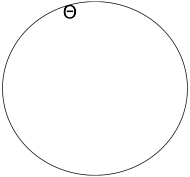
Estimation, upper bound: $P_\theta\left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta\right]$



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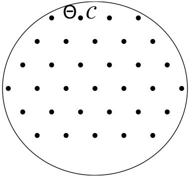

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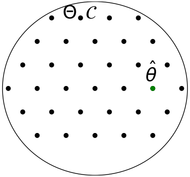

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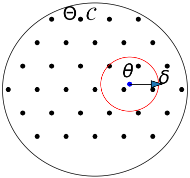
Estimator

$$\hat{\theta}(y^n) \triangleq argmax_{\theta' \in \mathcal{C}} \left[\frac{dP_{\theta'}}{d\mu}(y^n) \right]$$

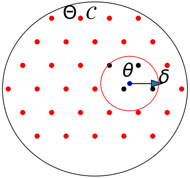
Error Probability Upper Bound - Basic Analysis

$$P_\theta\left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta\right] = ?$$



- θ is chosen arbitrarily (for minimax bound)

Error Probability Upper Bound - Basic Analysis



- Denote "bad" candidates $S \triangleq \{\theta' \in \mathcal{C} : \ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta\}$

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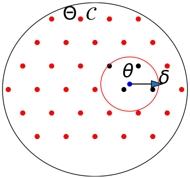

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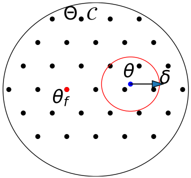

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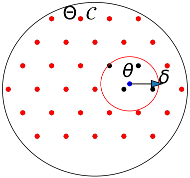
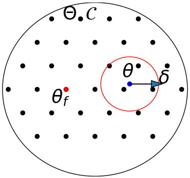
Basic Analysis: Candidate Count



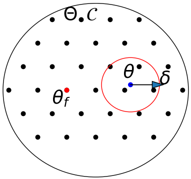
- Bound amount of far candidates by $|\mathcal{C}|$

Basic Analysis: Candidate Error Exponent



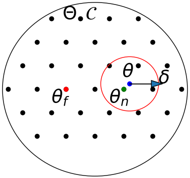
- Let far candidate $\theta_f \in \mathcal{C}$: $\ell(\theta, \theta_f) \geq \delta$.

Basic Analysis: Candidate Error Exponent



- Let far candidate $\theta_f \in \mathcal{C}$: $\ell(\theta, \theta_f) \geq \delta$.
- $P_\theta \left[\hat{\theta} = \theta_f \right] =??$

Basic Analysis: Candidate Error Exponent



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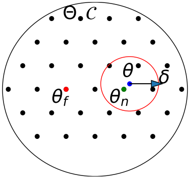

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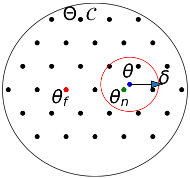
Basic Analysis: Candidate Error Exponent



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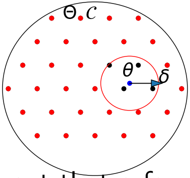
- $\log P_\theta \left[\hat{\theta} = \theta_f\right] \leq \log P_\theta \left[\frac{dP_{\theta_f}}{dP_{\theta_n}}(Y^n) \geq 1\right]$

Def: Hypothesis Testing Error Exponent (Mismatched,Worst Case)

$$E_{HT}(\tau, \delta) = \min_{\substack{\theta, \theta_n, \theta_f \in \Theta \\ \ell(\theta, \theta_n) \leq \tau \\ \ell(\theta, \theta_f) \geq \delta}} -\frac{1}{d} \log P_\theta \left[\frac{dP_{\theta_f}}{dP_{\theta_n}}(Y^n) \geq 1\right]$$

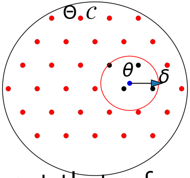
Basic Analysis: Global Error Exponent



- Upper bound on the event that a far candidate is the winner

$$\log P_\theta\left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta\right] \lessapprox -d\left[E_{HT}(\tau, \delta) - \frac{1}{d}\log|\mathcal{C}|\right]$$

Basic Analysis: Global Error Exponent



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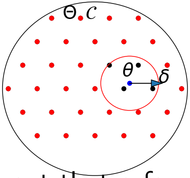
- "Competition": E_{HT} VS amount of candidates

Basic Analysis: Global Error Exponent



- Upper bound on the event that a far candidate is the winner

$$\log P_\theta\left[\ell\left(\theta,\hat{\theta}(Y^n)\right)>\delta\right]\lessapprox -d\left[E_{HT}(\tau,\delta)-\frac{1}{d}\log|\mathcal{C}|\right]$$

- "Competition": E_{HT} VS amount of candidates
- Large $\mathrm{Vol}(\Theta) \Rightarrow$ useless bound

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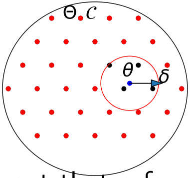

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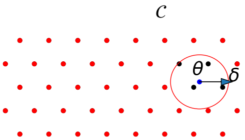
- "Competition": E_{HT} VS amount of candidates
- Large $\mathrm{Vol}(\Theta) \Rightarrow$ useless bound

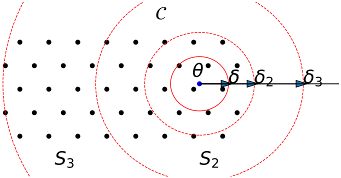
If P_θ and ℓ "matched": far candidates $\iff$ high error exponent
$\Rightarrow$ Can exploit this using a finer bounding method

Error Probability Upper Bound - Shell Analysis



- $P_\theta\left[\hat{\theta} \in S\right] = ?$

Error Probability Upper Bound - Shell Analysis



- $P_\theta\left[\hat{\theta} \in S\right] =?$
- Radii: $\delta = \delta_1 < \delta_2 < \cdots < \delta_k = \mathrm{diam}(\Theta)$

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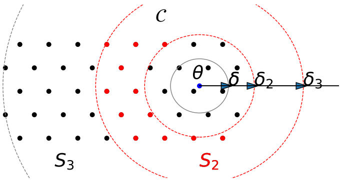

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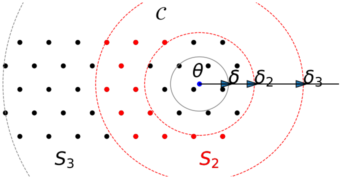
Error Probability Upper Bound - Shell Analysis



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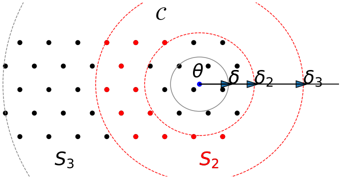

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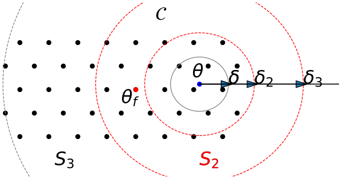

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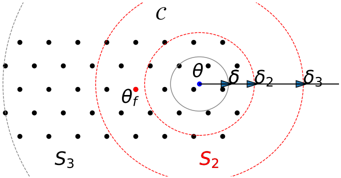


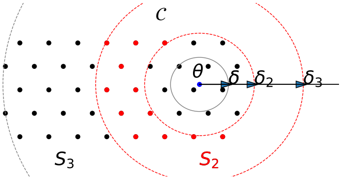
- $P_\theta\left[\hat{\theta} \in S_i\right] = \sum_{\theta_f \in S_i} P_\theta\left[\hat{\theta} = \theta_f\right]$
- $-\frac{1}{d}\log P_\theta\left[\hat{\theta} = \theta_f\right] \geq E_{HT}(\tau, \delta_i)$

Shell Analysis - Density Control

Should control candidate count

Def: Ball around θ with radius r w.r.t loss ℓ: $\mathcal{B}(\theta, r, \ell)$

$|S_i|$

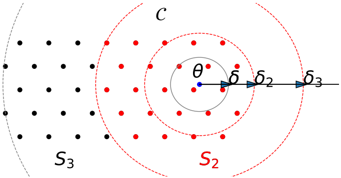


Shell Analysis - Density Control

Should control candidate count in balls

Def: Ball around θ with radius r w.r.t loss ℓ: $\mathcal{B}(\theta, r, \ell)$

$$|S_i| \leq |\mathcal{C} \cap \mathcal{B}(\theta, \delta_{i+1}, \ell)|$$

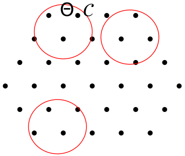


Shell Analysis - Density Control

Should control candidate count in balls for every center

Def: Ball around θ with radius r w.r.t loss ℓ: $\mathcal{B}(\theta, r, \ell)$

$$|S_i| \leq |\mathcal{C} \cap \mathcal{B}(\theta, \delta_{i+1}, \ell)| \leq max_{\theta' \in \Theta} |\mathcal{C} \cap \mathcal{B}(\theta', \delta_{i+1}, \ell)|$$



Shell Analysis - Density Control

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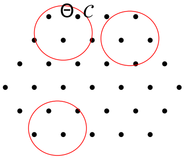
Def: Ball around θ with radius r w.r.t loss ℓ: $\mathcal{B}(\theta, r, \ell)$

$$\frac{1}{d}\log|S_i| \leq \frac{1}{d}\log\left(max_{\theta' \in \Theta}|\mathcal{C} \cap \mathcal{B}(\theta', \delta_{i+1}, \ell)|\right)$$



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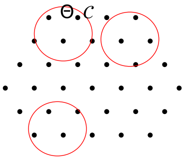

Def: Net Population Function

$$M_{\mathcal{C}}(r) = \frac{1}{d}\log\max_{\theta'\in\mathcal{C}}|\mathcal{C}\cap\mathcal{B}(\theta', r, \ell)|$$

Shell Analysis - Density Control

Should control candidate count in balls for every center, radius

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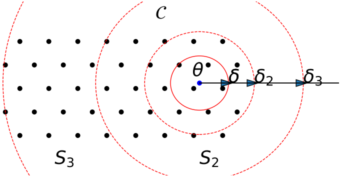

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Existence of τ-net with nearly optimal density

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Theorem

Let $d > 25$, $\Theta \subset \mathbb{R}^d$, and $\ell(\theta, \hat{\theta}) = \|\theta - \hat{\theta}\|$ for some arbitrary norm on $\mathbb{R}^d$. There exists a (lattice) τ-cover $\mathcal{C}$ of Θ satisfying

$$M_{\mathcal{C}}(\delta) \leq \log\left(\frac{\delta}{\tau}\right) + 3\frac{\log d}{d} + \frac{133 \log 2}{d}, \quad \forall \delta > \tau.$$

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Bound is tight: $\forall$ τ-cover $M_{\mathcal{C}}(\delta) \geq \log\left(\frac{\delta}{\tau}\right)$

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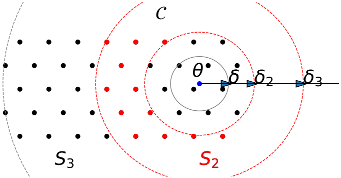
Bound is tight: $\forall$ τ-cover $M_{\mathcal{C}}(\delta) \geq \log\left(\frac{\delta}{\tau}\right)$

Proof is based on a uniform lattice covering result of O.-Regev-Weiss. Result of similar spirit can be derived using Erdős and Rogers '62

Shell Analysis - "Shell Error Exponent"

$$P_\theta\left[\hat{\theta} \in S_i\right] = \sum_{\theta_f \in S_i} P_\theta\left[\hat{\theta} = \theta_f\right]$$



"Shell error exponent":

$$\log P_\theta\left[\hat{\theta} \in S_i\right] \lessapprox -d\left[E_{HT}(\tau, \delta_i) - M_{\mathcal{C}}(\delta_{i+1})\right]$$

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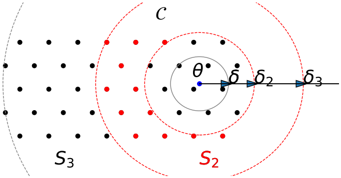
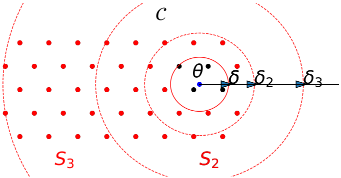


"Shell error exponent":

$$\log P_\theta\left[\hat{\theta} \in S_i\right] \lessapprox -d\left[E_{HT}(\tau, \delta_i) - \log\left(\frac{\delta_{i+1}}{\tau}\right) + o(1)\right]$$

Shell Analysis - Global Error Exponent

$$P_\theta\left[\hat{\theta} \in S\right] = \sum_{i=1}^{k} P_\theta\left[\hat{\theta} \in S_i\right]$$



If number of shells k is sub-exponential, P_e dictated by "Dominant shell error exponent":

$$\min_{i=1,\ldots,k-1}\left[E_{HT}(\tau,\delta_i) - \log\left(\frac{\delta_{i+1}}{\tau}\right) + o(1)\right] \overset{?}{>} 0$$

Shell Analysis: General Result

Theorem

Let

- $\mathcal{C}$ be a τ-cover with "good" density
- $\delta_i = \delta \cdot e^{\frac{i-1}{d}}$, $i = 1, \ldots, k = e^{o(d)}$
  note that $\log\left(\frac{\delta_{i+1}}{\tau}\right) = \log\left(\frac{\delta_i}{\tau}\right) + o(1)$

$$-\frac{1}{d} \log P_\theta \left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta\right] \geq \min_{i=1,\ldots,k-1} \left[E_{HT}(\tau, \delta_i) - \log\left(\frac{\delta_i}{\tau}\right) + o(1)\right]$$

Shell Analysis: General Result

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$$-\frac{1}{d}\log P_\theta\left[\ell\left(\theta, \hat{\theta}(Y^n)\right) > \delta\right]$$
$$\geq \min_{\delta' > \delta}\left[E_{HT}(\tau, \delta') - \log\left(\frac{\delta'}{\tau}\right) + o(1)\right]$$

Shell Analysis: General Result

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Upper bound $\iff$ Lower bound

- $\log\left(\frac{\delta'}{\tau}\right) \iff$ Rate distortion function of $\Theta, \ell = \|\theta - \hat{\theta}\|$
- $E_{HT}(\tau, \delta) \iff C(P_{Y^n|\theta})$

Main Result

Def: Critical Loss

$$\delta_*(\tau) \triangleq \sup \left\{ \delta' > 0 : \left[E_{HT}(\tau, \delta') - \log \frac{\delta'}{\tau} \right] \leq 0 \right\}$$

$$\delta_* \triangleq \min_{\tau > 0} \delta_*(\tau)$$

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Theorem

$$\min_{\hat{\theta}(\cdot)} \max_{\theta} P_\theta \left[\left\| \theta - \hat{\theta}(Y^n) \right\| > \delta_* \right] \xrightarrow{d \to \infty} 0$$

Main Result

Def: Critical Loss

$$\delta_*(\tau) \triangleq \sup\left\{\delta' > 0 : \left[E_{HT}(\tau,\delta') - \log\frac{\delta'}{\tau}\right] \leq 0\right\}$$

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Theorem

$$\min_{\hat{\theta}(\cdot)} \max_{\theta} P_\theta\left[\left\|\theta - \hat{\theta}(Y^n)\right\| > \delta_*\right] \xrightarrow{d\to\infty} 0$$

- To use theorem: bound $E_{HT}(\tau,\delta)$, choose τ

Error Exponent in Gaussian Cases + Examples

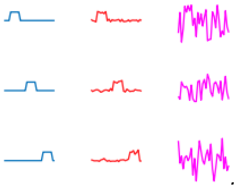
- $\ell(\theta,\theta') = \|\theta-\theta'\|_2^2$
- Transformation $g:\Theta \to \mathbb{R}^n$
- $P_\theta = \mathcal{N}(g(\theta), \sigma^2 I_n)$

Bound $E_{HT}(\tau,\delta) \geq \frac{1}{4}\psi(\tau,\delta)^2$

$$\psi(\tau,\delta) \triangleq \frac{1}{\sqrt{d}} \min_{\substack{\theta,\theta_n,\theta_f\in\Theta \\ \ell(\theta,\theta_n)\leq\tau \\ \ell(\theta,\theta_f)\geq\delta}} \left[\sqrt{\frac{\|g(\theta_f)-g(\theta)\|_2^2}{2\sigma^2}} - \sqrt{\frac{\|g(\theta_n)-g(\theta)\|_2^2}{2\sigma^2}}\right]$$

- Interpretation: Euclidean geometry optimization
- For GLM: $g(\theta)=\theta \to \delta^* < 32\log(2)\sigma^2 d$
- For spiked Wigner: $g(\theta) = \lambda \mathrm{vec}(\theta\theta^T)$, $\sigma^2 = \frac{1}{d}$ (we assume $\|\theta\| > 1$, $\lambda > \sqrt{58}$) $\to \delta^* < \frac{58}{\lambda^2}$

Example - Multi Reference Alignment



- $Y_j = R_{k_j}\theta + \sigma Z_j$, $j = 1, \ldots, m$, and $k_j \overset{i.i.d.}{\sim} \mathrm{Unif}([d])$
- Define extended parameter space $\tilde{\Theta} = \mathbb{R}^d \times [d]^m$, such that $\tilde{\theta} = (\theta, k_1, \ldots, k_m)$ also includes the (nuisance) shifts, and $\tilde{\ell}(\tilde{\theta}, \hat{\tilde{\theta}}) = \frac{1}{m}\|g(\tilde{\theta}) - g(\hat{\tilde{\theta}})\|_2^2$, where $g(\tilde{\theta}) = \mathrm{vec}(R_{k_1}\theta, \ldots, R_{k_m}\theta)$

$$\min_{\hat{\theta}(\cdot)} \max_{\theta < e^{e^{o(d)}}} P_\theta \left[\min_k \|R_k\theta - \hat{\theta}(Y^m)\|_2^2 \geq \frac{32\sigma^2 d}{m} \left(\log 2 + m \frac{\log d}{d} \right) \right] \to 0$$

- Upper bound equivalent to GLM for $m = O\left(\frac{d}{\log d}\right)$

Conclusions

- We presented a general framework to upper bound minimax risk (PAC setup), which is applicable to essentially unbounded parameter spaces
- The bound is based on a delicate shell analysis and mismatched BHT
- For $\Theta \subset \mathbb{R}^d, \ell(\theta, \theta') = \|\theta - \theta'\|$ our bound takes a relatively simple form
- For discrete product distributions on $\mathcal{Y}^n$, if $\frac{P_{\theta,j}(y)}{P_{\theta',j}(y)} \in 1 \pm \kappa_{n,d}$ for all $j \in [n], y \in \mathcal{Y}, \theta, \theta' \in \Theta$ we can prove that

$$E_{HT}(\tau, \delta) \geq \frac{1}{4d}\left[\sqrt{D_\delta} - \sqrt{D_\tau}\right]^2 + \frac{n}{d} \cdot O(\kappa_{n,d}^3),$$

where

$D_\tau \triangleq \sup_{\substack{\theta,\theta_n \in \Theta \\ \ell(\theta,\theta_n) \leq \tau}} D\left(P_\theta \| P_{\theta_n}\right), D_\delta \triangleq \inf_{\substack{\theta,\theta_n \in \Theta \\ \ell(\theta,\theta_n) \geq \delta}} D\left(P_\theta \| P_{\theta_f}\right)$