

# NEW BOUNDS ON THE DENSITY OF LATTICE COVERINGS

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ABSTRACT. We obtain new upper bounds on the minimal density  $\Theta_{n,\mathcal{K}}$  of lattice coverings of  $\mathbb{R}^n$  by dilates of a convex body  $\mathcal{K}$ . We also obtain bounds on the probability (with respect to the natural Haar-Siegel measure on the space of lattices) that a randomly chosen lattice  $L$  satisfies  $L + \mathcal{K} = \mathbb{R}^n$ . As a step in the proof, we utilize and strengthen results on the discrete Kakeya problem.

## 1. INTRODUCTION

The classical lattice covering problem asks for the most economical way to cover space by overlapping Euclidean balls centered at points of a lattice. To make this precise, given a lattice  $L \subset \mathbb{R}^n$ , normalized so that it has covolume one, define its *covering density*, denoted  $\Theta(L)$ , to be the minimal volume of a closed Euclidean ball  $B_r$ , for which  $\mathbb{R}^n = L + B_r$ . Define

$$\Theta_n \stackrel{\text{def}}{=} \inf\{\Theta(L) : L \text{ is a lattice of covolume one in } \mathbb{R}^n\}.$$

Similarly, let  $\mathcal{K} \in \text{Conv}_n$ , where  $\text{Conv}_n$  denotes the set of compact convex subsets of  $\mathbb{R}^n$  with nonempty interior. We define the  $\mathcal{K}$ -*covering density* of  $L$ , denoted  $\Theta_{\mathcal{K}}(L)$ , to be the minimal volume of a dilate  $r \cdot \mathcal{K}$  such that  $\mathbb{R}^n = L + r \cdot \mathcal{K}$ , and define

$$\Theta_{n,\mathcal{K}} \stackrel{\text{def}}{=} \inf\{\Theta_{\mathcal{K}}(L) : L \text{ is a lattice of covolume one in } \mathbb{R}^n\}.$$

The quantities  $\Theta_n$  and  $\Theta_{n,\mathcal{K}}$  have been intensively investigated, both for individual  $n$  and  $\mathcal{K}$ , and asymptotically for large  $n$ , and many questions remain open. Standard references are [CS88, GL87, Rog64].

The collection  $\mathcal{L}_n$  of lattices of covolume one in  $\mathbb{R}^n$  can be identified with the quotient  $\text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z})$ , via the map

$$g \text{SL}_n(\mathbb{Z}) \mapsto g\mathbb{Z}^n \quad (g \in \text{SL}_n(\mathbb{R})). \quad (1.1)$$

This identification endows  $\mathcal{L}_n$  with a natural probability measure; namely, there is a unique  $\text{SL}_n(\mathbb{R})$ -invariant Borel probability measure on  $\mathcal{L}_n$ . We will refer to this measure as the *Haar-Siegel measure* and denote it

by  $\mu_n$ . In this paper we give new bounds on  $\Theta_{n,\mathcal{K}}$  and on the  $\mu_n$ -typical value of  $\Theta_{\mathcal{K}}(L)$ .

**Theorem 1.1.** *There is  $c > 0$  so that for any  $n \in \mathbb{N}$  and any  $\mathcal{K} \in \text{Conv}_n$ ,*

$$\Theta_{n,\mathcal{K}} \leq cn^2. \quad (1.2)$$

This improves on the best previous bound of  $n^{\log_2 \log n + c}$ , which was proved by Rogers [Rog59]. We note that for the case that  $\mathcal{K}$  is the Euclidean ball, Rogers obtained  $\Theta_n \leq n(\log n)^c$  [Rog59], and this was extended to certain symmetric convex bodies by Gritzmann [Gri85]. This bound is better than what we obtain here.

We will actually prove the following measure estimate, from which Theorem 1.1 follows immediately.

**Theorem 1.2.** *There are positive constants  $c_1, c_2, c_3, c_4$  such that for any  $n \in \mathbb{N}$ , any  $\mathcal{K} \in \text{Conv}_n$ , and any*

$$M \in [c_3n^2, c_4n^3], \quad (1.3)$$

*we have*

$$\mu_n(\{L \in \mathcal{L}_n : \Theta_{\mathcal{K}}(L) > M\}) < c_1 e^{-\frac{c_2 M}{n^2}}. \quad (1.4)$$

We remark that the constants appearing in the statement of Theorem 1.2 can be explicitly estimated.

**Remark 1.3.** *As we will show in Appendix B, the left-hand side of (1.4) is at least  $C/M$ , for some constant  $C$  depending on  $n$  and  $\mathcal{K}$ . It follows that some upper bound on  $M$  is required if (1.4) is to hold. It also follows that the expectation of  $\Theta_{\mathcal{K}}$  with respect to the measure  $\mu_n$  is infinite.*

Setting  $M = c_4n^3$  in (1.4), we see that

**Corollary 1.4.** *There is a constant  $c > 0$  such that for any sequence  $\mathcal{K}_n \in \text{Conv}_n$ , the Haar-Siegel probability that  $\Theta_{n,\mathcal{K}_n}(L) \leq cn^3$  tends to 1 exponentially fast, as  $n \rightarrow \infty$ .*

This resolves a question of Strömbergsson, who showed in [Str12] that the conclusion holds with  $\Theta_{n,\mathcal{K}_n}(L) \leq (1 + \delta)^n$  and  $\delta > \delta_0$ , for an explicit number  $\delta_0 = 0.756\dots$

We introduce two quantities which describe the growth rate of the Haar-Siegel typical covering density. Let

$$\tau_o \stackrel{\text{def}}{=} \inf \{s > 0 : \mu_n \{L \in \mathcal{L}_n : \Theta(L) < n^s\} \xrightarrow{n \rightarrow \infty} 1\}$$

and

$$\tau \stackrel{\text{def}}{=} \inf \left\{ s > 0 : \inf_{\mathcal{K} \in \text{Conv}_n} \mu_n \{L \in \mathcal{L}_n : \Theta_{\mathcal{K}}(L) < n^s\} \xrightarrow{n \rightarrow \infty} 1 \right\}.$$

Clearly  $\tau_o \leq \tau$ , and a result of Coxeter, Few and Rogers [CFR59] implies that  $\tau_o \geq 1$ . Plugging  $M = n^{2+\varepsilon}$  into (1.4) we deduce the following.

**Corollary 1.5.** *We have  $\tau \leq 2$ .*

Prior to our results it was not known whether  $\tau$  and  $\tau_o$  are finite, i.e., whether the typical behavior of the covering density is polynomial. It would be interesting to know whether our upper bound on  $\tau$  and  $\tau_o$  can be improved.

**1.1. Simultaneous covering and packing.** We describe another application of Theorem 1.2, improving a result of Butler [But72]. To state it, define the  $\mathcal{K}$ -packing density of  $L$ , denoted  $\delta_{\mathcal{K}}(L)$ , to be the maximal volume of a dilate  $r \cdot \mathcal{K}$  such that the translates  $\{\ell + r \cdot \mathcal{K} : \ell \in L\}$  are disjoint. Then we have:

**Corollary 1.6.** *There is  $c > 0$  such that for all  $n \in \mathbb{N}$  and all  $\mathcal{K} \in \text{Conv}_n$ , there is  $L \in \mathcal{L}_n$  such that*

$$\frac{\Theta_{\mathcal{K}}(L)}{\delta_{\mathcal{K}}(L)} \leq c \frac{\text{Vol}(\mathcal{K} - \mathcal{K})}{\text{Vol}(\mathcal{K})} n^2. \tag{1.5}$$

This improves a previous upper bound of  $(\text{Vol}(\mathcal{K} - \mathcal{K})/\text{Vol}(\mathcal{K}))n^{\log_2 \log n + c}$  proved by Butler [But72]. The proof follows by observing that (a) a dilate of volume  $\ll \text{Vol}(\mathcal{K})/\text{Vol}(\mathcal{K} - \mathcal{K})$  is with high probability packing for a Haar-Siegel random  $L$  (by Siegel’s theorem [Sie45]), and that (b) a dilate of volume  $\gg n^2$  is with high probability covering for a Haar-Siegel random  $L$  (by Theorem 1.2). The union bound then shows that with high probability both events hold simultaneously, completing the proof. We leave the details to the reader.

The fact that (1.5) holds with high probability for a  $\mu_n$ -random lattice can be used to derive the following strengthening. Since  $\mu_n$  is preserved by the mapping which sends  $L$  to its dual  $L^*$  (see (B.2)), we obtain the existence of a lattice  $L$  such that both  $L$  and  $L^*$  satisfy (1.5).

**1.2. Ingredients of the proof.** Our proof of Theorem 1.2 utilizes some lower bounds on the cardinality of discrete Kakeya sets (see §2.4). Specifically, relying on a result of Kopparty, Lev, Saraf, and Sudan [KLSS11], we obtain a new lower bound on the size of a discrete  $\varepsilon$ -Kakeya set of rank 2, see Corollary 2.9. What is important for us is that the dependence of this bound on the parameter  $\varepsilon$  is linear.

We also use a variant of the Hecke correspondence to analyze the properties of a  $\mu_n$ -typical lattice. Namely, we show in §2.1 that for parameters  $p, r$ , if one draws a Haar-Siegel random lattice  $L$ , and then

replaces it by a lattice  $L'$  uniformly drawn from those containing  $L$  as a sub-lattice of index  $p^r$ , and with a prescribed quotient group  $L'/L$ , then  $L'$  (properly rescaled) is also Haar-Siegel random. Our construction is inspired by a similar construction which was investigated by Erez, Litsyn and Zamir [ELZ05] in the information theory literature.

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## 2. PRELIMINARIES

**2.1. Space of lattices and Haar-Siegel measure.** Recall from the introduction that  $\mathcal{L}_n \cong \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ . This space is endowed with the quotient topology and hence with the Borel  $\sigma$ -algebra arising from this topology. The group  $\mathrm{SL}_n(\mathbb{R})$  acts naturally on lattices via the linear action of matrices on  $\mathbb{R}^n$ , or equivalently, by left translations on the quotient  $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ . The measure  $\mu_n$  is the unique Borel probability measure on  $\mathcal{L}_n$  which is invariant under this action. From generalities on coset spaces of Lie groups, such a measure exists and is unique, see e.g., [Rag72]. We will also consider a slightly more general space, namely for each  $c > 0$  we write  $\mathcal{L}_{n,c}$  for the collection of lattices of covolume  $c$  in  $\mathbb{R}^n$ . The obvious rescaling isomorphism  $\mathcal{L}_n \cong \mathcal{L}_{n,c}$  commutes with the  $\mathrm{SL}_n(\mathbb{R})$ -action, and thus there is a unique  $\mathrm{SL}_n(\mathbb{R})$ -invariant measure on  $\mathcal{L}_{n,c}$ , and we will denote it by  $\mu_{n,c}$ . We will refer to any of the measures  $\mu_n, \mu_{n,c}$  as the Haar-Siegel measure.

For a prime  $p$  and an integer  $r \in \{1, \dots, n\}$ , associate to each lattice  $L$  the finite collection  $\Lambda_{p,r}(L)$  of lattices  $L'$  in  $\mathbb{R}^n$  which contain  $L$  as a sub-lattice, and for which the quotient  $L'/L$  is isomorphic to  $\prod_1^r \mathbb{Z}/p\mathbb{Z}$ . Note that these lattices are of covolume  $p^{-r}$ . The assignment  $L \rightarrow \Lambda_{p,r}(L)$  is a particular case of the so-called *Hecke correspondence* (see e.g., [COU01]). The following useful observation is well-known, we include a proof for completeness.

**Proposition 2.1.** *For each  $n, p, r$  as above, let  $N = |\Lambda_{p,r}(\mathbb{Z}^n)|$ . Then for each  $f \in C_c(\mathcal{L}_{n,p^{-r}})$ , i.e., each continuous compactly supported real valued function on  $\mathcal{L}_{n,p^{-r}}$ ,*

$$\int f d\mu_{n,p^{-r}} = \int_{\mathcal{L}_n} \frac{1}{N} \sum_{L' \in \Lambda_{p,r}(L)} f(L') d\mu_n(L). \quad (2.1)$$

In other words, choosing  $L'$  randomly according to Haar-Siegel measure on  $\mathcal{L}_{n,p^{-r}}$  is the same as choosing  $L$  randomly according to Haar-Siegel measure on  $\mathcal{L}_n$ , and then choosing  $L'$  uniformly in  $\Lambda_{p,r}(L)$ .

*Proof.* The right-hand side of (2.1) describes a positive continuous linear functional on  $C_c(\mathcal{L}_{n,p^{-r}})$ , and hence, by the Riesz representation theorem, is equal to  $\int f d\nu$  for some Radon measure  $\nu$  on  $\mathcal{L}_{n,p^{-r}}$ . Taking a monotone increasing sequence of compactly supported functions tending everywhere to 1, we see that (2.1) holds for the function  $f \equiv 1$ , and from this it follows that  $\nu$  is a probability measure. By the uniqueness property of Haar-Siegel measure, in order to show that  $\nu = \mu_{n,p^{-r}}$  it suffices to show that  $\nu$  is invariant under left-multiplication by any  $g \in \mathrm{SL}_n(\mathbb{R})$ . From the definition of  $\Lambda_{p,r}(L)$  we see that  $\Lambda_{p,r}(gL) = g\Lambda_{p,r}(L)$ , and so the invariance of  $\nu$  follows from the following computation:

$$\begin{aligned} \int f \circ g d\nu &= \int_{\mathcal{L}_n} \frac{1}{N} \sum_{L' \in \Lambda_{p,r}(L)} f(gL') d\mu_n(L) \\ &= \int_{\mathcal{L}_n} \frac{1}{N} \sum_{L'' \in \Lambda_{p,r}(gL)} f(L'') d\mu_n(L) \\ &= \int_{\mathcal{L}_n} \frac{1}{N} \sum_{L'' \in \Lambda_{p,r}(gL)} f(L'') d\mu_n(gL) = \int f d\nu. \end{aligned}$$

□

We now interpret this in terms of the discrete Grassmannian, as follows. For a prime  $p$  let  $\mathbb{F}_p$  denote the field with  $p$  elements. For  $r \in \{1, \dots, n\}$ , let  $\mathrm{Gr}_{n,r}(\mathbb{F}_p)$  denote the collection of subspaces of dimension  $r$  in  $\mathbb{F}_p^n$ , or equivalently, the rank- $r$  additive subgroups of  $\mathbb{F}_p^n$ . We can identify  $\mathbb{F}_p$  with the residues  $\{0, \dots, p-1\}$ , and thus identify  $\mathbb{F}_p^n$  with the quotient  $\mathbb{Z}^n/p\mathbb{Z}^n$ . We have a natural *reduction mod  $p$*  homomorphism  $\pi_p : \mathbb{Z}^n \rightarrow \mathbb{F}_p^n$ , which sends each coordinate of  $x \in \mathbb{Z}^n$  to its class modulo  $p$ . Any element  $S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)$  gives rise to a sub-lattice  $\pi_p^{-1}(S) \subset \mathbb{Z}^n$ , which contains  $p\mathbb{Z}^n$  as a subgroup of index  $p^r$ , and with  $\pi_p^{-1}(S)/p\mathbb{Z}^n$  isomorphic as an abelian group to  $S \cong \prod_1^r \mathbb{Z}/p\mathbb{Z}$ . Similarly, for any  $L' \in \Lambda_{p,r}(\mathbb{Z}^n)$  we have  $S = \pi_p(pL') \cong L'/\mathbb{Z}^n$ . This shows that for any lattice  $L = g\mathbb{Z}^n$  we have

$$\Lambda_{p,r}(L) = \{p^{-1}g\pi_p^{-1}(S) : S \in \mathrm{Gr}_{n,r}(\mathbb{F}_p)\}.$$

We have shown:

**Proposition 2.2.** *Choosing  $L'$  according to  $\mu_{n,p^{-r}}$  is the same as choosing  $L = g\mathbb{Z}^n$  according to  $\mu_n$ , then choosing  $S \in \text{Gr}_{n,r}(\mathbb{F}_p)$  uniformly and setting  $L' = p^{-1}g\pi_p^{-1}(S)$ .*

We can state Proposition 2.2 in more concrete terms as follows. Choose a random lattice  $L$  distributed according to  $\mu_n$ , choose generators  $v_1, \dots, v_n$  of  $L$ , so that the parallelepiped

$$\mathcal{P}_L = \left\{ \sum a_i v_i : \forall i, 0 \leq a_i < 1 \right\} \quad (2.2)$$

is a fundamental domain for  $\mathbb{R}^n/L$ . Define the discrete ‘net’

$$\mathcal{P}_L^{(\text{disc})} \stackrel{\text{def}}{=} \left\{ \sum a_i v_i \in \mathcal{P}_L : a_i \in \left\{ 0, \frac{1}{p}, \dots, 1 - \frac{1}{p} \right\} \right\}. \quad (2.3)$$

These are coset representatives for the inclusion  $L \subset \frac{1}{p} \cdot L$ . Choose elements  $w_1, \dots, w_r \in \mathcal{P}_L^{(\text{disc})}$  from the uniform distribution over linearly independent (as elements of  $\mathbb{F}_p^n$ )  $r$ -tuples. Then the lattice  $L' = \text{span}_{\mathbb{Z}}(v_1, \dots, v_n, w_1, \dots, w_r)$  is a random lattice distributed according to  $\mu_{n,p^{-r}}$ .

**2.2. Some bounds of Rogers and Schmidt.** We now recall some fundamental results of Rogers and Schmidt. For a lattice  $L \in \mathcal{L}_n$  let  $\mathbb{T}_L \stackrel{\text{def}}{=} \mathbb{R}^n/L$  be the quotient torus, let  $m_L$  be the Haar probability measure on  $\mathbb{T}_L$ , and let  $\pi_L : \mathbb{R}^n \rightarrow \mathbb{T}_L$  be the quotient map. Let  $\text{Vol}(\cdot)$  denote the Lebesgue measure on  $\mathbb{R}^n$ . For a Borel measurable subset  $J \subset \mathbb{R}^n$ , and a lattice  $L \subset \mathbb{R}^n$ , let

$$\varepsilon(J, L) \stackrel{\text{def}}{=} 1 - m_L(\pi_L(J));$$

equivalently,  $\varepsilon(J, L)$  is the density of points in  $\mathbb{R}^n$  not covered by  $L + J$ . Also let

$$\eta = \eta_n \stackrel{\text{def}}{=} \frac{n}{4} \log \left( \frac{27}{16} \right) - 3 \log n. \quad (2.4)$$

With these notations, the following was shown in [Rog58] (see also [Sch58]):

**Theorem 2.3.** *There is a positive constant  $c_{\text{Rog}}$  such that for all  $n \in \mathbb{N}$ , for every Borel measurable  $J \subset \mathbb{R}^n$  with*

$$V \stackrel{\text{def}}{=} \text{Vol}(J) \leq \eta$$

we have

$$\left| \int_{\mathcal{L}_n} \varepsilon(J, L) d\mu_n(L) - e^{-V} \right| < c_{\text{Rog}} \cdot e^{-\eta}.$$

Using the Markov inequality, this immediately implies the following:

**Corollary 2.4.** *With the same notation and assumptions, for any  $\kappa > 0$ ,*

$$\mu_n(\{L \in \mathcal{L}_n : \varepsilon(J, L) > \kappa\}) < \frac{1}{\kappa} (e^{-V} + c_{\text{Rog}} e^{-n}). \quad (2.5)$$

**2.3. From half covering to full covering.** Here we show the standard fact (cf. [Rog59, Lemma 4] or [HLR09]) that if a convex body covers half the space, then dilating it by a factor 2 covers all of space. Notice that this translates to a factor  $2^n$  in volume, as a result of which we will only use this lemma for very small bodies.

**Lemma 2.5.** *Let  $\mathcal{K} \in \text{Conv}_n$  and let  $L$  be a lattice in  $\mathbb{R}^n$ . Suppose that*

$$m_L(\pi_L(\mathcal{K})) > \frac{1}{2}.$$

*Then we have*

$$L + 2\mathcal{K} = \mathbb{R}^n.$$

*Proof.* Since

$$m_L(\pi_L(\mathcal{K})) = m_L(\pi_L(-\mathcal{K})) > \frac{1}{2},$$

we have that for any  $x \in \mathbb{T}_L$ ,

$$m_L((\pi_L(\mathcal{K}) - x) \cap (\pi_L(-\mathcal{K}))) > 0.$$

Therefore, there are  $z_1, z_2 \in \pi_L(\mathcal{K})$  so that  $z_1 - x = -z_2$ , or equivalently, there are  $y_1, y_2 \in \mathcal{K}$  so that

$$x = \pi_L(y_1) + \pi_L(y_2) = \pi_L(y_1 + y_2).$$

The claim now follows from  $y_1 + y_2 \in 2\mathcal{K}$ .  $\square$

**2.4. Lower bound on the size of a discrete  $\varepsilon$ -Kakeya set.** Now let  $q$  be a power of a prime, let  $\mathbb{F}_q$  denote the field with  $q$  elements and for a line  $\ell \in \text{Gr}_{n,1}(\mathbb{F}_q)$ , let  $x + \ell$  denote the affine line through  $x$  parallel to  $\ell$ . A subset  $K \subset \mathbb{F}_q^n$  is called a *Kakeya set* if for every  $\ell \in \text{Gr}_{n,1}(\mathbb{F}_q)$  there is  $x \in \mathbb{F}_q^n$  such that  $x + \ell \subset K$ ; that is,  $K$  contains a line in every direction. For  $\varepsilon \in (0, 1]$ ,  $K$  is called an  $\varepsilon$ -Kakeya set if

$$|\{\ell \in \text{Gr}_{n,1}(\mathbb{F}_q) : \exists x \text{ s.t. } x + \ell \subset K\}| \geq \varepsilon |\text{Gr}_{n,1}(\mathbb{F}_q)|;$$

that is  $K$  contains a line in at least an  $\varepsilon$ -proportion of directions. Extending this notion to higher dimensions, let  $\varepsilon \in (0, 1]$  and  $r \in \{1, \dots, n-1\}$ . Then a set  $K \subset \mathbb{F}_q^n$  is called a *Kakeya set of rank  $r$*  if for any  $S \in \text{Gr}_{n,r}(\mathbb{F}_q)$  there is  $x \in \mathbb{F}_q^n$  such that  $x + S \subset K$ , and an  $\varepsilon$ -Kakeya set of rank  $r$  if

$$|\{S \in \text{Gr}_{n,r}(\mathbb{F}_q) : \exists x \text{ s.t. } x + S \subset K\}| \geq \varepsilon |\text{Gr}_{n,r}(\mathbb{F}_q)|.$$

In this subsection we will derive lower bounds on the size of an  $\varepsilon$ -Kakeya set of rank  $r$ . Our main observation is that the possible sizes of an  $\varepsilon$ -Kakeya set and a  $\delta$ -Kakeya set are related as follows.

**Lemma 2.6.** *Let  $0 < \varepsilon < \delta < 1$ . Assume  $K \subset \mathbb{F}_q^n$  is an  $\varepsilon$ -Kakeya set of rank  $r$ , then there exists a  $\delta$ -Kakeya set  $\mathcal{A} \subset \mathbb{F}_q^n$  of rank  $r$  with cardinality*

$$|\mathcal{A}| \leq \left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil |K|.$$

*Proof.* Fix  $n \in \mathbb{N}$  and  $r \in \{1, \dots, n-1\}$ . For  $K \subset \mathbb{F}_q^n$ , denote

$$\mathcal{B}_K \stackrel{\text{def}}{=} \{S \in \text{Gr}_{n,r}(\mathbb{F}_q) : K \text{ contains a translate of } S\}.$$

Let  $g$  be an element of  $\text{GL}_n(\mathbb{F}_q)$ , that is an invertible  $n \times n$  matrix with entries in  $\mathbb{F}_q$ . For  $S \in \text{Gr}_{n,r}(\mathbb{F}_q)$ , we clearly have  $S \in \mathcal{B}_K$  if and only if  $gS \in \mathcal{B}_{gK}$ . Let  $\mathcal{N}$  be a finite subset of  $\text{GL}_n(\mathbb{F}_q)$  and consider

$$\mathcal{A} = \mathcal{A}(\mathcal{N}, K) \stackrel{\text{def}}{=} \bigcup_{g \in \mathcal{N}} gK.$$

Clearly

$$|\mathcal{A}| \leq |\mathcal{N}| \cdot |K| \quad \text{and} \quad \bigcup_{g \in \mathcal{N}} g\mathcal{B}_K \subset \mathcal{B}_{\mathcal{A}}.$$

Recall that by definition of an  $\varepsilon$ -Kakeya set of rank  $r$ , we have that  $|\mathcal{B}_K| \geq \varepsilon |\text{Gr}_{n,r}(\mathbb{F}_q)|$ . Consequently, our claim will follow once we show that if

$$\mathcal{B} \subset \text{Gr}_{n,r}(\mathbb{F}_q) \text{ satisfies } |\mathcal{B}| \geq \varepsilon |\text{Gr}_{n,r}(\mathbb{F}_q)| \quad (2.6)$$

then

$$\exists \mathcal{N} \subset \text{GL}_n(\mathbb{F}_q) \text{ s.t. } |\mathcal{N}| \leq \left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil \quad \text{and} \quad \left| \bigcup_{g \in \mathcal{N}} g\mathcal{B} \right| \geq \delta |\text{Gr}_{n,r}(\mathbb{F}_q)|. \quad (2.7)$$

We will prove this using a standard probabilistic argument. Define a probability space by drawing  $N = \left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil$  elements  $g_1, \dots, g_N$  of  $\text{GL}_n(\mathbb{F}_q)$ , uniformly and independently. Fix  $\mathcal{B}$  as in (2.6) and for each  $S \in \text{Gr}_{n,r}(\mathbb{F}_q)$ , denote by  $E_S^i$  the event that  $S \notin g_i\mathcal{B}$ . The events  $\{E_S^i : i = 1, \dots, N\}$  are i.i.d., since the  $g_i$  are. Therefore

$$E_S \stackrel{\text{def}}{=} \bigcap_{i=1}^N E_S^i,$$

satisfies

$$\Pr(E_S) = (\Pr(E_S^i))^N.$$



Since  $\mathrm{GL}_n(\mathbb{F}_q)$  acts transitively on  $\mathrm{Gr}_{n,r}(\mathbb{F}_q)$ ,

$$\Pr(E_S^i) = \Pr(g_i^{-1}S \notin \mathcal{B}) = 1 - \frac{|\mathcal{B}|}{|\mathrm{Gr}_{n,r}(\mathbb{F}_q)|} \leq 1 - \varepsilon,$$

which implies

$$\Pr(E_S) \leq (1 - \varepsilon)^N \leq 1 - \delta. \quad (2.8)$$

It therefore follows that

$$\mathbb{E} \left| \bigcup_{i=1}^N g_i \mathcal{B} \right| = \sum_{S \in \mathrm{Gr}_{n,r}(\mathbb{F}_q)} (1 - \Pr(E_S)) \geq \delta |\mathrm{Gr}_{n,r}(\mathbb{F}_q)|.$$

This implies that there exists a subset  $\mathcal{N} \stackrel{\mathrm{def}}{=} \{g_1, \dots, g_N\}$  that satisfies (2.7).  $\square$

In [Dvi09, DKSS13, KLSS11], a fundamental lower bound on the minimal cardinality of Kakeya sets was established. We will need the following variant, whose special case  $\delta = 1$  was proved in [KLSS11]:

**Lemma 2.7.** *Let  $\delta \in (0, 1]$ . If  $K \subset \mathbb{F}_q^n$  is a  $\delta$ -Kakeya set of rank  $r$  then*

$$|K| \geq \left(1 + \frac{(q-1)q^{-r}}{\delta}\right)^{-n} q^n.$$

The proof follows with minor adaptations from the arguments of [KLSS11]. We give the details in Appendix A.

The bound in Lemma 2.7 is quite tight for large  $\delta$ , but is loose for  $\delta \ll 1$ . We now leverage Lemma 2.6 to obtain a much sharper bound for small  $\delta$ . It replaces the exponential (in  $n$ , with  $q, r$  fixed) dependence on  $\delta$  with a linear dependence. We remark that the bound (2.10) will not be used in this paper, and is included for future reference.

**Theorem 2.8.** *Let  $\varepsilon \in (0, 1)$ . If  $K \subset \mathbb{F}_q^n$  is an  $\varepsilon$ -Kakeya set of rank  $r$ , then*

$$|K| > \varepsilon (1 + 2(q-1)q^{-r})^{-n} q^n, \quad (2.9)$$

and for  $r = 1$  we also have that

$$|K| > \varepsilon \frac{e^{-1}}{\log(2en)} 2^{-n} q^n. \quad (2.10)$$

*Proof.* We first claim that if  $K \subset \mathbb{F}_q^n$  is an  $\varepsilon$ -Kakeya set of rank  $r$ , then for any  $\varepsilon < \delta < 1$

$$|K| \geq \left( \left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil \right)^{-1} \cdot \left(1 + \frac{1}{\delta}(q-1)q^{-r}\right)^{-n} q^n. \quad (2.11)$$

To see this, let  $\varepsilon < \delta < 1$ , and assume for contradiction that  $K \subset \mathbb{F}_q^n$  is an  $\varepsilon$ -Kakeya set of rank  $r$  with cardinality smaller than the right-hand

side of (2.11). By Lemma 2.6, this implies that there must exist a  $\delta$ -Kakeya set  $\mathcal{A} \in \mathbb{F}_q^n$  of rank  $r$  with cardinality  $|\mathcal{A}| \leq \left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil |K| < (1 + \frac{1}{\delta}(q-1)q^{-r})^{-n} q^n$ , which contradicts Lemma 2.7.

Next, we use (2.11) to show that if  $|K|$  is an  $\varepsilon$ -Kakeya set of rank  $r$ , it must satisfy (2.9). For  $\varepsilon \in [1/2, 1)$ , this follows immediately from Lemma 2.7. We may therefore assume without loss of generality that  $\varepsilon \in (0, 1/2)$ . Let  $\delta = 1/2$  and note that for all  $\varepsilon$  in this range

$$\left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil = \left\lceil \frac{\log(2)}{-\log(1-\varepsilon)} \right\rceil < \frac{\log(2)}{-\log(1-\varepsilon)} + 1 < \frac{1}{\varepsilon}.$$

Thus, applying (2.11) with  $0 < \varepsilon < \delta = 1/2$  establishes (2.9).

Finally, we assume  $r = 1$  and establish (2.10). Let  $\delta = \frac{1}{1+2/n}$  and note that

$$\left(1 + \frac{1}{\delta}(q-1)q^{-r}\right)^{-n} q^n \geq e^{-1} 2^{-n} q^n.$$

Hence, for  $\varepsilon \in [\delta, 1]$ , (2.10) follows immediately from Lemma 2.7. We may therefore assume without loss of generality that  $\varepsilon \in (0, \delta)$ . For all  $\varepsilon$  in this range

$$\left\lceil \frac{\log(1-\delta)}{\log(1-\varepsilon)} \right\rceil = \left\lceil \frac{\log(1 + \frac{n}{2})}{-\log(1-\varepsilon)} \right\rceil < \frac{\log(2n)}{\varepsilon} + 1 < \frac{\log(2en)}{\varepsilon}.$$

Thus, applying (2.11) with  $0 < \varepsilon < \delta = \frac{1}{1+2/n}$  establishes (2.10).  $\square$

We will need the following consequence:

**Corollary 2.9.** (i) *If  $K \subset \mathbb{F}_q^n$  is an  $\varepsilon$ -Kakeya set of rank 2 then  $\frac{|K|}{q^n} > \varepsilon e^{-2n/q}$ .*

(ii) *If  $K' \subset \mathbb{F}_q^n$  satisfies  $\frac{|K'|}{q^n} \geq 1 - \varepsilon e^{-2n/q}$  then the set*

$$\mathcal{S} \stackrel{\text{def}}{=} \{S \in \text{Gr}_{n,2}(\mathbb{F}_q) : \forall x \in \mathbb{F}_q^n, (x+S) \cap K' \neq \emptyset\}$$

*satisfies*

$$|\mathcal{S}| > (1-\varepsilon)|\text{Gr}_{n,2}(\mathbb{F}_q)|.$$

*Proof.* For  $r = 2$  we have that

$$\frac{1}{(1+2(q-1)q^{-r})^{-n}} = \left(1 + \frac{2}{q} - \frac{2}{q^2}\right)^n \leq \left[\left(1 + \frac{2}{q}\right)^{q/2}\right]^{2n/q} \leq e^{2n/q}.$$

Thus (i) is an immediate consequence of (2.9). Assertion (ii) follows from (i) by setting  $K = \mathbb{F}_q^n \setminus K'$ .  $\square$

## 3. PROOF OF THEOREM 1.2

Let  $p$  be a prime number satisfying

$$n \leq p \leq 2n. \quad (3.1)$$

We define a probability space as follows. Let  $L = g\mathbb{Z}^n$  be a random lattice chosen according to  $\mu_n$ , and  $S$  be randomly chosen from the uniform distribution on  $\text{Gr}_{n,2}(\mathbb{F}_p)$ , independently of  $L$ . Define the lattice  $L' = p^{-1}g\pi_p^{-1}(S)$  and note that  $L \subset L' \subset \frac{1}{p}L$ . By Proposition 2.2, we have that  $L'$  is distributed according to  $\mu_{n,p^{-2}}$ . Therefore, the left-hand side of (1.4), which we are trying to bound from above, is equal to

$$\Pr\left(\Theta_{\mathcal{K}}(L) > M\right) = \Pr\left(\Theta_{\mathcal{K}}(L') > \frac{M}{p^2}\right).$$

Let  $J$  be the dilate of  $\mathcal{K}$  of volume

$$V \stackrel{\text{def}}{=} p^{-2} \left(1 + \frac{2}{p}\right)^{-n} M. \quad (3.2)$$

Applying Corollary 2.4 with  $\kappa = e^{-\frac{V}{2}}$  we have

$$\Pr\left(\varepsilon(J, L) > e^{-\frac{V}{2}}\right) < c_0 e^{-\frac{V}{2}},$$

where  $c_0 \stackrel{\text{def}}{=} 1 + c_{\text{Rog}}$ . Here we used that  $V \leq \eta$  (where  $\eta$  is as defined in (2.4)) which holds assuming the constant  $c_4$  is chosen small enough. From now on, we fix an  $L$  for which

$$\varepsilon(J, L) \leq e^{-\frac{V}{2}}, \quad (3.3)$$

and we show that when choosing  $S$ , with probability at least  $1 - \varepsilon$ , for  $\varepsilon$  to be chosen below, we have  $\Theta_{\mathcal{K}}(L') \leq M/p^2$ .

Define

$$B_L \stackrel{\text{def}}{=} \mathbb{T}_L \setminus \pi_L(J), \quad (3.4)$$

so that  $m_L(B_L) \leq e^{-V/2}$ . Let  $\mathcal{P}_L^{(\text{disc})}$  be as in (2.3), and let

$$\overline{\mathcal{P}}_L^{(\text{disc})} \stackrel{\text{def}}{=} \pi_L\left(\mathcal{P}_L^{(\text{disc})}\right) = \pi_L\left(\frac{1}{p} \cdot L\right) \subset \mathbb{T}_L. \quad (3.5)$$

The Haar measure  $m_L$  on the torus  $\mathbb{T}_L$  satisfies that

$$m_L(A) = \frac{1}{p^n} \sum_{x \in \overline{\mathcal{P}}_L^{(\text{disc})}} m_L(A - x),$$

and applying this with  $A$  taken to be  $B_L$  we find that there is  $u \in \mathbb{T}_L$  such that

$$\frac{1}{p^n} \left| \left(u + \overline{\mathcal{P}}_L^{(\text{disc})}\right) \cap B_L \right| \leq m_L(B_L) \leq e^{-\frac{V}{2}}.$$

Recall that we have an identification of  $\mathbb{F}_p^n$  with  $(0, \frac{1}{p}, \dots, 1 - \frac{1}{p})^n$  by reducing mod  $p$  and then dividing by  $p$ , and a further identification of  $(0, \frac{1}{p}, \dots, 1 - \frac{1}{p})^n$  with  $\overline{\mathcal{P}}_L^{(\text{disc})}$ . With these identifications in mind we view  $\mathbb{F}_p^n$  as a subset of  $\mathbb{T}_L$ , and define

$$K' \stackrel{\text{def}}{=} \{x \in \mathbb{F}_p^n : u + x \in \pi_L(J)\},$$

so that

$$\frac{|K'|}{p^n} \geq 1 - e^{-\frac{V}{2}}.$$

This implies via Corollary 2.9(ii), applied with

$$\varepsilon \stackrel{\text{def}}{=} e^{-\frac{V}{2}} e^{2n/p}, \quad (3.6)$$

that with probability at least  $1 - \varepsilon$  over the choice of  $S$ , it holds that for all  $x \in \mathbb{F}_p^n$ ,  $u + x + S$  intersects  $\pi_L(J)$ . Recalling that  $L' = \frac{1}{p} \cdot g\pi_p^{-1}(S)$ , this equivalently says that

$$u + \frac{1}{p} \cdot L \subset L' + J. \quad (3.7)$$

But by Lemma 2.5 and (3.3), and using that  $V > 2 \log 2$  (which we can assume by taking  $c_3$  large enough), we have

$$L + 2J = \mathbb{R}^n.$$

Together with (3.7), this implies that

$$L' + \left(1 + \frac{2}{p}\right)J \supset u + \frac{1}{p} \cdot L + \frac{2}{p}J = \mathbb{R}^n.$$

To summarize, Proposition 2.2 shows that with all but probability  $c_0 e^{-V/2} + \varepsilon$  (due to the choice of  $L$  and  $S$ ), we have  $L' + \left(1 + \frac{2}{p}\right)J = \mathbb{R}^n$  and hence  $\Theta_{\mathcal{K}}(L') \leq \frac{M}{p^2}$ . Using our choices (3.1), (3.2) and (3.6) we see that for appropriate choices of constants  $c_1, c_2$ , we have (1.4).

## APPENDIX A. PROOF OF LEMMA 2.7

The case  $\delta = 1$  is precisely [KLSS11, Theorem 1]. The general case  $\delta \in (0, 1]$  (as in Lemma 2.7) follows from minor modifications to their proof. For the reader's convenience, we include the full proof here, much of it taken verbatim from [KLSS11].

We start with some necessary background. Let  $\mathbb{N}_0$  denote the set of non-negative integers. For an  $n$ -tuple  $i = (i_1, \dots, i_n) \in \mathbb{N}_0^n$ , we define  $\|i\| \stackrel{\text{def}}{=} i_1 + \dots + i_n$  and if  $X = (X_1, \dots, X_n)$  then  $X^i \stackrel{\text{def}}{=} X_1^{i_1} \dots X_n^{i_n}$ . Any

polynomial  $P$  in  $n$  variables over some field  $\mathbb{F}$  can be expanded in the form

$$P(X + Y) = \sum_{i \in \mathbb{N}_0^n} P^{(i)}(Y)X^i ,$$

for some polynomials  $P^{(i)}$  over  $\mathbb{F}$  in  $n$  variables. We refer to  $P^{(i)}$  as the *Hasse derivative of  $P$  of order  $i$* . It is easy to see that  $P^{(0)} = P$  and that for  $\|i\| > \deg P$ ,  $P^{(i)} = 0$ . Moreover, if  $P_H$  denotes the homogeneous part of  $P$ , then

$$(P_H)^{(i)} = \begin{cases} (P^{(i)})_H & \text{if } \deg P^{(i)} = \deg P - \|i\|, \\ 0 & \text{if } \deg P^{(i)} < \deg P - \|i\|. \end{cases}$$

For a nonzero polynomial  $P$  in  $n$  variables over a field  $\mathbb{F}$ , we define its *multiplicity of zero* at some point  $a \in \mathbb{F}^n$ , denoted  $\mu(P, a)$ , as the largest  $m \geq 0$  such that  $P^{(i)}(a) = 0$  for all  $i \in \mathbb{N}_0^n$  with  $\|i\| < m$ . Alternatively, it is the largest  $m$  for which we can write

$$P(X + a) = \sum_{i \in \mathbb{N}_0^n : \|i\| \geq m} c(i, a)X^i$$

for some  $c(i, a) \in \mathbb{F}$ . We sometimes also say that  $P$  *vanishes* at  $a$  with multiplicity  $m$ .

We will use the following relatively straightforward lemmas.

**Lemma A.1** ([DKSS13, Lemma 5]). *Let  $n \geq 1$  be an integer. For any nonzero polynomial  $P$  in  $n$  variables over a field  $\mathbb{F}$ ,  $a \in \mathbb{F}^n$ , and  $i \in \mathbb{N}_0^n$ , it holds that*

$$\mu(P^{(i)}, a) \geq \mu(P, a) - \|i\| .$$

**Lemma A.2** ([DKSS13, Proposition 10]). *Let  $n, m \geq 1$  and  $k \geq 0$  be integers, and  $\mathbb{F}$  a field. If a finite set  $S \subset \mathbb{F}^n$  satisfies  $\binom{m+n-1}{n}|S| < \binom{n+k}{n}$ , then there exists a nonzero polynomial over  $\mathbb{F}$  in  $n$  variables of degree at most  $k$ , vanishing at every point of  $S$  with multiplicity at least  $m$ .*

**Lemma A.3** ([KLSS11, Lemma 14]). *Let  $n, r \geq 1$  be integers, and  $P$  a nonzero polynomial in  $n$  variables over a field  $\mathbb{F}$ . Suppose that  $b, d_1, \dots, d_r \in \mathbb{F}^n$ . Then for any  $t_1, \dots, t_r \in \mathbb{F}$ ,*

$$\mu(P(b + T_1 d_1 + \dots + T_r d_r), (t_1, \dots, t_r)) \geq \mu(P, b + t_1 d_1 + \dots + t_r d_r) ,$$

where we view  $P(b + T_1 d_1 + \dots + T_r d_r)$  as a polynomial in the formal variables  $T_1, \dots, T_r$ .

Finally, we will need a multiplicity version of the standard Schwartz-Zippel lemma [DKSS13].

**Lemma A.4** ([DKSS13, Lemma 2.7]). *Let  $n \geq 1$  be an integer,  $P$  a nonzero polynomial in  $n$  variables over a field  $\mathbb{F}$ , and  $S \subset \mathbb{F}$  a finite set. Then*

$$|S|^{-(n-1)} \sum_{z \in S^n} \mu(P, z) \leq \deg P .$$

*Proof of Lemma 2.7.* Let  $m, k$  be positive integers satisfying

$$k < \delta q^r \left\lceil \frac{qm - k}{q - 1} \right\rceil . \quad (\text{A.1})$$

Our goal for the rest of the proof is to show that under the condition (A.1),

$$|K| \geq \binom{n+k}{n} / \binom{m+n-1}{n} . \quad (\text{A.2})$$

The lemma then follows by taking  $k = Nq^{r+1} - 1$  and  $m = \lceil (q^r + \frac{q-1}{\delta})N \rceil$  where  $N$  is a positive integer. With this choice, (A.1) holds, and the lemma follows by noting that the right-hand side of A.2 converges to  $(1 + (q-1)q^{-r}/\delta)^{-n} q^n$  as  $N$  goes to infinity.

Assume towards contradiction that (A.2) does not hold. Thus, by Lemma A.2, there exists a nonzero polynomial  $P$  in  $n$  variables over  $\mathbb{F}_q$  of degree at most  $k$  that vanishes at every point of  $K$  with multiplicity at least  $m$ . Let  $\ell \stackrel{\text{def}}{=} \lceil \frac{qm-k}{q-1} \rceil$  and fix  $i = (i_1, \dots, i_n) \in \mathbb{N}_0^n$  satisfying  $w \stackrel{\text{def}}{=} \|i\| < \ell$ . Let  $Q \stackrel{\text{def}}{=} P^{(i)}$  be the  $i$ th Hasse derivative of  $P$ .

Let  $D \subset (\mathbb{F}_q^n)^r$  be the set of all  $r$ -tuples of vectors  $(d_1, \dots, d_r)$  with the property that there exists  $b \in \mathbb{F}_q^n$  such that  $b + t_1 d_1 + \dots + t_r d_r \in K$  for all  $t_1, \dots, t_r \in \mathbb{F}_q$ . Since  $K$  is a  $\delta$ -Kakeya set of rank  $r$ , we have  $|D| \geq \delta \cdot q^{nr}$ . (Notice that the span of  $d_1, \dots, d_r$  might be of rank less than  $r$ ; the statement is true because a  $\delta$ -Kakeya set of rank  $r$  is also a  $\delta$ -Kakeya set of rank  $r'$  for all  $r' \leq r$ .) Therefore, by our choice of  $P$ , for any  $(d_1, \dots, d_r) \in D$ , there exists a  $b \in \mathbb{F}_q^n$  such that for all  $t_1, \dots, t_k \in \mathbb{F}_q$ ,

$$\mu(P, b + t_1 d_1 + \dots + t_r d_r) \geq m ,$$

and so by Lemma A.3 and Lemma A.1,

$$\begin{aligned} \mu(Q(b + T_1 d_1 + \dots + T_r d_r), (t_1, \dots, t_r)) &\geq \mu(Q, b + t_1 d_1 + \dots + t_r d_r) \\ &\geq m - w , \end{aligned}$$

where in the left-hand side we consider  $Q(b + T_1 d_1 + \dots + T_r d_r)$  as a polynomial in the variables  $T_1, \dots, T_r$ . But since

$$\deg Q(b + T_1 d_1 + \dots + T_r d_r) \leq \deg Q \leq k - w < q(m - w)$$

(which follows from  $w < \ell$ ), Lemma A.4 with  $S = \mathbb{F}_q$  implies that  $Q(b + T_1 d_1 + \dots + T_r d_r)$  is in fact the zero polynomial.

Let  $P_H$  and  $Q_H$  denote the homogeneous parts of  $P$  and  $Q$ , respectively (i.e.,  $P_H$  is the unique homogeneous polynomial for which  $\deg(P - P_H) < \deg P$ ). It is easy to see that  $Q(b + T_1 d_1 + \dots + T_r d_r) = 0$  implies  $Q_H(T_1 d_1 + \dots + T_r d_r) = 0$  (note that there is no  $b$  in the latter). It follows that  $(P_H)^{(i)}(T_1 d_1 + \dots + T_r d_r) = 0$  for all  $(d_1, \dots, d_r) \in D$ . Equivalently,  $(P_H)^{(i)}$ , considered as a polynomial in  $n$  variables over the field of rational functions  $\mathbb{F}_q(T_1, \dots, T_r)$ , vanishes at every point of the set

$$D' \stackrel{\text{def}}{=} \{T_1 d_1 + \dots + T_r d_r : (d_1, \dots, d_r) \in D\} \subset S^n,$$

where

$$S \stackrel{\text{def}}{=} \{\alpha_1 T_1 + \dots + \alpha_r T_r : \alpha_1, \dots, \alpha_r \in \mathbb{F}_q\}.$$

Since  $i$  is an arbitrary tuple satisfying  $\|i\| < \ell$ , this shows that  $P_H$  vanishes with multiplicity at least  $\ell$  at every point of  $D'$ . On the other hand, by (A.1),

$$\deg P_H = \deg P \leq k < \delta q^r \ell = \delta |S| \ell,$$

which implies by Lemma A.4 that  $P_H$  is the zero polynomial. This is a contradiction since the homogeneous part of a nonzero polynomial is nonzero.  $\square$

#### APPENDIX B. THE EXPECTATION OF THE COVERING DENSITY IS INFINITE

**Proposition B.1.** *There is  $c > 0$  such that for all  $n$  large enough and all  $M \geq 1$  we have*

$$\mu_n(\{L \in \mathcal{L}_n : \Theta(L) > M\}) \geq \frac{c V_n^2}{2^n} \frac{1}{M}, \quad (\text{B.1})$$

where  $V_n$  denotes the volume of the Euclidean ball of radius one in  $\mathbb{R}^n$ . In particular, for any  $n$  and any  $\mathcal{K} \in \text{Conv}_n$  there is  $C > 0$  such that

$$\mu_n(\{L \in \mathcal{L}_n : \Theta_{\mathcal{K}}(L) > M\}) > \frac{C}{M}.$$

*Proof.* Let  $\lambda_1(L)$  denote the length of the shortest nonzero vector of  $L$ . Given  $M$ , let  $r \stackrel{\text{def}}{=} (M/V_n)^{1/n}$  be the radius of a Euclidean ball of volume  $M$ . Let  $L^*$  denote the dual lattice of  $L$ , that is

$$L^* \stackrel{\text{def}}{=} \{u \in \mathbb{R}^n : \forall v \in L, u \cdot v \in \mathbb{Z}\}. \quad (\text{B.2})$$

By considering the distance between affine hyperplanes perpendicular to the shortest nonzero vector of  $L^*$ , we see that  $\lambda_1(L^*) < \frac{1}{2r}$  implies

that  $\Theta(L) > M$ . In particular, taking into account that the measure  $\mu_n$  is invariant under the mapping  $\mathcal{L}_n \rightarrow \mathcal{L}_n$ ,  $L \mapsto L^*$ , we see that the left-hand side of (B.1) is bounded below by

$$\mu_n \left( \left\{ L \in \mathcal{L}_n : \lambda_1(L) < \frac{1}{2r} \right\} \right). \quad (\text{B.3})$$

Using Siegel's summation formula, Kleinbock and Margulis [KM99, §7] obtained the estimate

$$\mu_n(\{L \in \mathcal{L}_n : \lambda_1(L) < t\}) \geq \frac{1}{2\zeta(n)} V_n t^n - \frac{1}{4\zeta(n-1)\zeta(n)} V_n^2 t^{2n},$$

where  $\zeta(n) = \sum_{m \in \mathbb{N}} m^{-n}$  is the Riemann zeta function. Applying this estimate with  $t = \frac{1}{2r}$  and using  $M = V_n r^n$  and the fact that  $\zeta(n) \rightarrow_{n \rightarrow \infty} 1$ , we get that the left-hand side of (B.1) is bounded from below by

$$\frac{V_n^2}{2^{n+2}M} - \frac{V_n^4}{2^{2n+2}M^2}.$$

By standard estimates for  $V_n$ , the first summand in this expression is the dominant one for  $M \geq 1$ .

This proves (B.1). Since any  $\mathcal{K} \in \text{Conv}_n$  is contained in a dilate of a Euclidean ball, the second assertion of the proposition follows.  $\square$

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